FACE VECTORS OF SUBDIVISIONS OF BALLS

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A fundamental invariant of a subdivision of a space into cells is its collection of face numbers or f-vector. A major area of study is understanding the possible f-vectors of various types of subdivisions. Most of this thesis works on characterizing the f-vectors of subdivisions of balls. We study two different types of subdivisions: simplicial complexes that are triangulations of balls and simplicial posets whose order complexes are balls.

For simplicial complexes we describe two methods for showing that a vector cannot be the f-vector of a homology d-ball. As a consequence, we disprove a conjectured characterization of the f-vectors of triangulated balls of dimension five and higher due to Billera and Lee. We also provide a construction of triangulated balls with various f-vectors. We show that this construction obtains all possible f-vectors of triangulated three- and four-balls and we conjecture that this result also extends to dimension five.

For simplicial posets we use Stanley's idea of the face ring to develop a series of new conditions on the f-vectors of simplicial poset balls. We also present new methods for constructing simplicial poset balls with specific f-vectors. Combining this work with a result of Murai we give a complete characterization of the f-vectors of simplicial poset balls in even dimensions, as well as odd dimensions less than or equal to five.

The last chapter looks at surgeries of triangulated manifolds. First we derive formulas for the face numbers of a particular triangulation of the product of two simplicial complexes. Using these formulas, we determine how some surgeries change the face numbers of a manifold. This leads to new results about the f-vectors of certain manifolds.

BIOGRAPHICAL SKETCH

Samuel Kolins was born in Detroit, Michigan, on September 28, 1984. He grew up in the Detroit area and attended Birmingham Seaholm High School. He did his undergraduate work at Bowdoin College in Brunswick, Maine, where he majored in mathematics and physics. Sam then entered the Ph.D. program at Cornell University in Ithaca, New York. On the verge of going into analysis during his second year, a course he took from Ed Swartz enticed him to study a mix of combinatorics, commutative algebra, and topology. After graduation, Samuel will be an assistant professor at Lebanon Valley College in Annville, Pennsylvania.

In memory of Len Kolins.

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Chapter 1

Introduction

1.1 Subdivisions of Topological Spaces

A major area of study in combinatorics is looking at the properties of various types of subdivisions of a topological space. In this thesis we encounter four major types of subdivisions. Chapter 2 focuses mainly on triangulations into simplicial complexes, while Chapter 3 deals with simplicial cell decompositions (also know as simplicial posets). In a couple of instances we also discuss polyhedral balls and the boundaries of simplicial polytopes. These are special types of subdivisions of balls and spheres respectively. We begin by defining these various kinds of subdivisions.

An (abstract) *simplicial complex* Δ on vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ that is closed under inclusion and contains all of the one-element sets $\{i\}$ for $i \in [n]$. The elements of Δ are called *faces*, and the *dimension of a face* $F \in \Delta$ is $\dim F := |F| - 1$. The *dimension of* Δ is equal to the maximum of the dimensions of all of its faces and is usually be denoted $d-1$. Given an abstract simplicial complex Δ, a *geometric realization of* Δ, $|\Delta|$, is a collection of Euclidean simplexes $\{\sigma_F : F \in \Delta - \emptyset\}$ such that

- $\dim \sigma_F = \dim F = |F| - 1$,
- if $G \subseteq F$ then σ_G is a face of σ_F,
- $\sigma_{F \cap G} = \sigma_F \cap \sigma_G$.

Parts of Chapter 2 were published in *Discrete & Computational Geometry* [15] and parts of Chapter 3 have been accepted for publication in the *Journal of Algebraic Combinatorics* [16].

One can show that all geometric realizations of Δ are homeomorphic, so we are justified in using the notation $|\Delta|$ for any geometric realization of Δ. We say that Δ is *homeomorphic* to a topological space X if $|\Delta|$ is homeomorphic to X. A *triangulation* of a topological space X is a simplicial complex that is homeomorphic to X. Throughout the following we use the same notation to refer to both a subset $S \in \Delta$ and the corresponding simplex in a geometric realization, with the meaning clear from the context.

In a few situations we study the particular type of triangulations of spheres given by the boundaries of simplicial polytopes. In general, a *polytope* P is the convex hull of a finite set of points in $\mathbb{R}^d$. The boundary ∂P of a polytope P has a natural cell complex structure. The faces of this structure are given by the intersections of ∂P with the hyperplanes that do not intersect the relative interior of P. If all of these faces are simplices then we call the polytope simplicial. In this case there exists a simplicial complex Δ such that ∂P is a geometric realization of Δ. Since ∂P is topologically a sphere, Δ is a triangulation of a sphere. Note that if all of the vertices of a polytope are in general position, then the polytope is simplicial.

While the boundary of a simplicial polytope is always a sphere, we can construct a ball by deleting an open disk from our sphere. We define a *polyhedral ball* to be a triangulated ball obtained from the boundary of a simplicial polytope by removing a single vertex and all faces containing this vertex.

Given any regular CW-complex C, the *face poset* of C, denoted $P(C)$, is the poset with elements the closed faces (or cells) of C ordered by inclusion. The empty face is the minimal element $\hat{0}$ of the face poset. In the special case where C is a single $(d-1)$-simplex, its face poset is called the *Boolean algebra of rank d*. Equivalently, the Boolean algebra of rank d is the poset of all subsets of $[d]$ ordered by inclusion.

A *simplicial poset* P is a finite poset containing a minimal element $\hat{0}$ such that for every $p \in P$ the closed interval $[\hat{0}, p]$ is a Boolean algebra. Note that the face poset of any simplicial complex is a simplicial poset. There are simplicial posets that are not the face poset of any simplicial complex. However, for any simplicial poset P there is a regular CW-complex $\Gamma(P)$ such that P is the face poset of $\Gamma(P)$ (see [5]). The closed faces of $\Gamma(P)$ are topological simplexes, but two faces can intersect in a subcomplex of their boundaries instead of just a single face. In particular, $\Gamma(P)$ can have multiple faces on the same vertex set. Throughout this thesis we identify each closed face of $\Gamma(P)$ with the corresponding element of the poset P.

A *simplicial cell decomposition* of a topological space X is a simplicial poset P such that $\Gamma(P)$ is homeomorphic to X. For example, taking two triangles and gluing them along their entire boundaries in a natural way results in a simplicial cell decomposition of the 2-sphere that is not a triangulation of the 2-sphere. If P is a simplicial cell decomposition of a space X we often refer to properties of X as being properties of P. Hence if P is a simplicial cell decomposition of a ball we say that P is a simplicial poset ball.

1.2 Background on Counting Faces

Given a subdivision of a space, an important combinatorial invariant is its face vector or f-vector. The f-vector is defined as $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where f_i is the number of dimension i faces of the subdivision and $d-1$ is the dimension of the space. Here $f_{-1} = 1$, corresponding to the empty face. A fundamental question is to understand the relationships between the f-vector of a subdivision and topological properties of the underlying space. The most well known classical result of this type is the Euler characteristic of a triangulation of a manifold, which relates the alternating sum of the manifold's f-vector entries to a topological invariant of the manifold.

Intense interest in this area began in the middle of the 20th century with the question of characterizing the possible f-vectors of subdivisions of spheres, in particular the boundaries of polytopes. In 1971, McMullen conjectured a complete characterization of the possible f-vectors of the boundaries of simplicial polytopes [22]. This conjecture was verified in the early 1980's by Billera and Lee [4] and Stanley [32]. The g-conjecture (Conjecture 2.2) asserts that this characterization also holds in the more general setting of triangulations of spheres (this claim is actually one of many different versions of the "g-conjecture"). Results of Stanley [33] and Masuda [21] finished in 2005 completely characterize the f-vectors of simplicial cell decompositions of spheres.

In addition to spheres, recent work has been done on characterizing the f-vectors of subdivisions of other manifolds. For some low dimensional manifolds, complete characterizations of the possible f-vectors of triangulations are known (see for example [7], [36], [37]). Additional work has lead to inequalities between the Betti numbers of spaces and the entries of the f-vectors of triangulations. (see for example [6], [28], [18]). For simplicial cell decompositions, Murai recently characterized the f-vectors of subdivisions of $\mathbb{R}P^n$ and products of spheres [25].

In this thesis we first work on the problem of characterizing the possible f-vectors of two different types of subdivisions of balls. In Chapter 2 we examine simplicial complexes that are triangulations of (homology) balls, while in Chapter 3 we look at the same question for simplicial cell decompositions of balls. Finally, in Chapter 4 we study Cartesian products of simplicial complexes and related surgeries of triangulated manifolds. Using these surgeries, for some manifolds we are able to derive new bounds relating the manifold's f-vector and its Betti numbers.

1.3 Triangulations of Balls

In [3], Billera and Lee calculate a set of conditions on the f-vectors of triangulated balls that would follow from the g-conjecture. Billera and Lee conjecture that these conditions are not only necessary but also sufficient for a characterization of the f-vectors of balls (see Conjecture 2.7). Recently, Lee and Schmidt confirmed this conjecture for three- and four-dimensional balls [17].

In Section 2.3 we present two methods that show that certain vectors are not the f-vectors of triangulated balls. As a consequence, we show that the Billera and Lee conditions are not sufficient in dimensions five and higher. In both approaches, we assume that a ball with a certain f-vector exists and then show that there must be some way to split the ball along a codimension-one face to create two new balls. For some f-vectors we can show that the new balls created by this splitting cannot exist. The first technique relates one of the Betti numbers of the face ring of the ball to the existence of a codimension-one face along which we can split the ball. This has the advantage of being relatively straightforward to compute in particular examples. In the second method we look at all possible one-skeletons of a ball with a given f-vector and show that in each case the desired type of splitting is possible. This method generates an infinite class of counterexamples to the Billera and Lee conjecture in every dimension greater than four.

Section 2.4 presents a construction of triangulated balls with prescribed f-vectors. In Section 2.5 we show that these same f-vectors can be obtained using shellable balls, but this requires a more involved construction. In dimensions three and four these results duplicate the work of Lee and Schmidt in obtaining all possible f-vectors of triangulated balls. For the dimension five case, we conjecture that this construction gives all possible f-vectors. However, in dimensions higher than five we know that not all f-vectors of balls can be obtained with this approach, and we do not even have a conjectural description of the possible f-vectors.

1.4 Simplicial Poset Balls

Unlike the case of triangulations of balls, there is no previous literature on the question of characterizing the f-vectors of simplicial poset balls. In Chapter 3 we come up with a partial answer to this question. In Section 3.3 we relate the f-vector of a simplicial poset ball to the f-vector of its boundary poset, which is a sphere. This allows us to translate the known conditions on the f-vectors of spheres to conditions on our ball. However, as in the case of simplicial *complexes* that triangulate balls, these conditions are not sufficient to completely characterize the f-vectors of simplicial cell decompositions of balls.

In Sections 3.4 and 3.5 we develop additional new necessary conditions on the f-vectors of balls. These results use the algebraic idea of the face ring of a simplicial poset, introduced by Stanley [33]. The conditions are derived by looking at relationships between the face ring of the ball and the face rings of certain spheres, such as the boundary of the ball or the sphere obtained by adding to the ball the cone over the boundary of the ball. We use known results about the face rings of these spheres to obtain new conditions on the ball.

In Section 3.6 we present constructions for obtaining simplicial cell decompositions of balls with specific f-vectors. Section 3.7 combines the previous work with a new result of Murai to give a complete characterization of the f-vectors of simplicial posets that are balls in all even dimensions as well as in dimensions three and five.

1.5 Products of Simplicial Complexes and Surgery

We define the *ith Betti number* β_i of a manifold M with respect to a field k to be the vector space dimension of the ith reduced homology group of M, $\beta_i := \dim_k \tilde{H}_i(M; k)$. An active area of research is looking at the relationships between the f-vector and the Betti numbers of a triangulated manifold. A possible tool for studying this relationship is the idea of surgery on a manifold. To perform a surgery on a manifold of dimension $j + k - 1$, in the manifold we find (or create) a Cartesian product $B^k \times S^{j-1}$ of a triangulated ball and sphere. The boundary of this product is $\partial B^k \times S^{j-1}$, the product of a $(k-1)$-sphere and a $(j-1)$-sphere. Let B^j be a triangulated ball with $\partial B^j = S^{j-1}$. Then $\partial B^k \times B^j$ also has boundary $\partial B^k \times S^{j-1}$. Therefore, we can remove (the interior of) the product $B^k \times S^{j-1}$ from the original manifold and then glue in the product $\partial B^k \times B^j$ along the newly created boundary. An example of this type of surgery is handle addition; take two identical balls, i.e. $B^k \times S^0$, remove them from a manifold, and then glue in the 'handle' $\partial B^k \times \Delta^1$ (where Δ^1 is an interval) along the two holes $\partial B^k \times S^0$.

These surgeries provide a structured way to alter both the Betti numbers and the f-vector of a triangulated manifold. The most useful way to express the change in the face numbers of the manifold turns out to be in terms of the g-vector. The entries of the g-vector, denoted g_i, are completely determined by the dimension and the face numbers of the triangulation (see Section 2.1.2 for the definition). Theorem 4.3 shows that when $j > k$ the change in g_{k+1} of a $(d-1)$-manifold when $\partial B^k \times \Delta^j$ is replaced by $B^k \times \partial\Delta^j$ is $\binom{d+1}{k+1}$. If instead we replace $B^j \times \partial\Delta^k$ by $\partial B^j \times \Delta^k$, then Theorem 4.4 describes the more complicated behavior that can occur. Depending on how the product $\partial B^k \times B^j$ lies in the original manifold, these surgeries may also increase the kth Betti number of the manifold by one. For example, in the case $k = 1$ we are performing handle addition which adds one to the first Betti number of our manifold.

For triangulated manifolds, Kalai conjectured a lower bound on g_{k+1} of the form $g_{k+1} \geq \binom{d+1}{k+1}\beta_k + C$ where C depends on the Betti numbers β_i with $i < k$ (see Conjecture 4.5 and [12, Conjecture 14.2]). If the surgery described in Theorem 4.3 adds one to β_k and $\binom{d+1}{k+1}$ to g_{k+1}, it transforms a manifold that obtains this lower bound into a new manifold that also achieves the bound. Using the surgery of Theorem 4.4 it is possible to increase β_k by one but change g_{k+1} by less than $\binom{d+1}{k+1}$. In cases where Kalai's conjecture is known to be valid, the existence of this type of surgery allows us to place a new, stronger lower bound on g_{k+1}. We give an example of this in Section 4.4.

Chapter 2

f-vectors of Triangulated Balls

2.1 Notation and Background

We begin by discussing some needed background on simplicial complexes, the face ring, and commutative algebra. Stanley's book [35] is a good reference for most of the material in this section.

2.1.1 Basics of Simplicial Complexes

First we introduce some additional terminology related to simplicial complexes. Let Δ be a simplicial complex of dimension $d-1$ on vertex set $[n]$. A *facet* of Δ is any maximal face with respect to inclusion. A simplicial complex is *pure* if all of its facets have the same dimension. Any triangulation of a $(d-1)$-manifold is an example of a pure simplicial complex where all of the facets have dimension $(d-1)$.

In some cases, instead of thinking about the subsets of $[n]$ that are faces of Δ it is useful to focus on the subsets that are not faces. A *non-face* of Δ is a subset of $[n]$ that is not in Δ and a *non-i-face* of Δ is an $(i+1)$-subset of $[n]$ that is not an element of Δ. Note that if you know all of the non-faces of Δ this is equivalent to knowing the faces of Δ. In fact, all you need to know are the minimal non-faces (those not containing any strictly smaller non-face).

We also often work with two important types of subsets of Δ, induced subcomplexes and links. For $W \subset [n]$, $\Delta_W := \{\sigma \in \Delta : \sigma \subset W\}$ is the *induced subcomplex* on the vertex set W.

If $F \in \Delta$ then the *link of F in Δ* is $\mathrm{lk}_\Delta(F) := \{G \in \Delta : G \cup F \in \Delta, G \cap F = \emptyset\}$. If Δ is a pure simplicial complex of dimension $d-1$ then the link of F is a pure simplicial complex of dimension $d - \dim(F) - 2$.

2.1.2 The h-vector and Shellings

We now formalize our notation for counting the faces of a $(d-1)$-dimensional simplicial complex Δ. The *ith face number* of Δ, denoted $f_i(\Delta)$, is the number of i-dimensional faces of Δ. When it is clear to what complex we are referring we often just write f_i for the face numbers. So $f_{-1} = 1$ (corresponding to the empty set), $f_0 = n$, and $f_i = 0$ for $i \geq d$. The *f-vector* of Δ is the list of the face numbers, $f(\Delta) := (f_{-1}, f_0, f_1, \ldots, f_{d-1})$. The *$h$-vector* of Δ, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ contains the same combinatorial information as the f-vector but is often easier to use. Its entries are defined from the face numbers by

$$\sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i} \qquad \text{or} \qquad h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

In addition to the f- and h-numbers we also use the g-numbers defined by $g_i(\Delta) := h_i(\Delta) - h_{i-1}(\Delta)$ for $i \geq 1$ and $g_0(\Delta) = 1$. The *g-vector* of Δ is $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$. However, sometimes it is useful for us to consider g_i where $i > \lfloor d/2 \rfloor$.

Intitally, the f-vector seems like the natural object to consider, while the definition of the h-vector may appear a bit non-intuitive. However, there are many good reasons to work in terms of the h-numbers rather than the face-numbers. First, many results and symmetries are much easier to express in term of the h-vector. For example, the Dehn-Sommerville equations state that for triangulated spheres, $h_i = h_{d-i}$. The Euler characteristic of a simplicial complex is also easy to read off from the h-vector. If Δ is a

$(d-1)$-dimensional simplicial complex then

$$h_d = \sum_{i=0}^{d} (-1)^{d-i} f_{i-1} = (-1)^{d-1} \tilde{\chi}(|\Delta|).$$

In particular, if Δ is a homology $(d-1)$-ball then $h_d = 0$.

More deeply, the entries of the h-vector often have nice combinatorial and algebraic interpretations. We discuss the algebraic approach in Section 2.1.4 when we talk about the face ring. For the combinatorial viewpoint we now look at shellable complexes.

The idea of a shelling is to build our simplicial complex facet by facet such that each time we add a new facet it intersects the union of all the previous facets in a "nice" way. More formally, a *shelling order* of a pure simplicial complex Δ is an ordering of the facets of Δ, $F_1, \ldots, F_{f_{d-1}}$, such that for $j = 1, \ldots, f_{d-1}$, when F_j is added to $\cup_{i=1}^{j-1} F_i$ there is a unique new face that is minimal with respect to inclusion. This face is called the *restriction face* and is denoted $r(F_j)$. A complex is *shellable* if there exists a shelling order of its facets. We can obtain the h-vector of a shellable complex from the restriction faces of any shelling order by $h_i = |\{F_j : |r(F_j)| = i\}|$ [35, Proposition 2.3].

We can also give an equivalent definition of a shelling order in terms of the geometric realization of Δ. An ordering $F_1, \ldots, F_{f_{d-1}}$ of the (closed) facets of $|\Delta|$ is a shelling order if for all $j \geq 2$ the intersection $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is a non-empty union of (closed) *facets* of ∂F_j. In this case, the vertices of the restriction face $r(F_j)$ are the vertices of F_j opposite the facets of ∂F_j that are in the intersection $F_j \cap (\cup_{i=1}^{j-1} F_i)$.

2.1.3 Homology Manifolds

In some of our work, instead of considering triangulated balls it is useful to consider the larger class of homology balls. All of our homology is taken with coefficients in the integers. A pure simplicial complex Δ of dimension $d-1$ is a *homology* $(d-1)$*-manifold* if for every non-empty face $F \in \Delta$ the link of F has the same homology as the $(d-1-|F|)$-sphere or the $(d-1-|F|)$-ball. The *boundary* of a homology $(d-1)$-manifold Δ is defined to be $\partial\Delta := \{F \in \Delta : H_{d-1-|F|}(\mathrm{lk}_\Delta(F)) = 0\}$. From [24] we know that the boundary of a homology $(d-1)$-manifold is either empty or a homology $(d-2)$-manifold without boundary. A *homology* $(d-1)$*-sphere* is a homology $(d-1)$-manifold with empty boundary and the same homology as the $(d-1)$-sphere. A *homology* $(d-1)$*-ball* is a homology $(d-1)$-manifold with the same homology as the $(d-1)$-ball and boundary a homology $(d-2)$-sphere. From the long exact sequence of the homology of the pair $(\Delta, \partial\Delta)$, for a homology $(d-1)$-ball Δ we have $H_{d-1}(\Delta, \partial\Delta) \cong \mathbb{Z}$.

2.1.4 The Face Ring

We now introduce Stanley's powerful idea of the face ring of a simplicial complex. The face ring allows us to use tools from commutative algebra to study the h-vectors of simplicial complexes.

Let k be an infinite field of arbitrary characteristic and $R := k[x_1, \dots, x_n]$. For a simplicial complex Δ on vertex set $[n]$ the *face ring* (or *Stanley-Reisner ring*) $k[\Delta]$ is obtained by taking the quotient of R by the ideal generated by the monomials corresponding to *non-faces* of Δ. More specifically, $k[\Delta] := R/I_\Delta$ where

$$I_\Delta := (x_{i_1}x_{i_2}\cdots x_{i_k} : i_1 < i_2 < \cdots < i_k, \{i_1, i_2, \dots, i_k\} \notin \Delta).$$

A simplicial complex Δ is called *Cohen-Macaulay* if its face ring $k[\Delta]$ is Cohen-Macaulay. We will not go into the details of Cohen-Macaulay rings here. Instead, we use a result of Reisner [30] that Cohen-Macaulay complexes can be characterized by looking at the homology of the links of the faces in the complex.

Theorem 2.1 (Reisner, '76). *A simplicial complex Δ is Cohen-Macaulay (over k) iff for all faces $\sigma \in \Delta$*

$$\tilde{H}_i(\mathrm{lk}_\Delta \sigma; k) = 0 \qquad \textit{for all} \qquad i < \dim(\mathrm{lk}_\Delta \sigma).$$

As a consequence of this result, all homology balls and spheres are Cohen-Macaulay.

Still assuming $\dim \Delta = d-1$, a *linear system of parameters* (l.s.o.p.) for $k[\Delta]$ is a collection of degree one elements $\theta_1, \ldots, \theta_d \in k[\Delta]$ such that $k[\Delta]/(\theta_1, \ldots, \theta_d)$ is finite-dimensional over k. For an infinite field k a generic choice of d degree one elements of $k[\Delta]$ is an l.s.o.p. Given an l.s.o.p. $\theta_1, \ldots, \theta_d$ of Δ, define $k(\Delta) := k[\Delta]/(\theta_1, \ldots, \theta_d)$. Although $k(\Delta)$ does depend on the choice of an l.s.o.p., all of our results involving $k(\Delta)$ hold for any l.s.o.p. so we use the notation $k(\Delta)$ without specifying an l.s.o.p.

Let T be a graded ring such that $T = R/I$ for some homogeneous ideal I. Denote by T_i the ith homogeneous component of T. The *Hilbert function* of T is given by $F(T, i) := \dim_k T_i$. If Δ is a Cohen-Macaulay complex, then $F(k(\Delta), i) = h_i$. This gives us an algebraic interpretation of the h-numbers of homology balls and spheres.

2.1.5 Algebraic Betti Numbers and Hochster's Formula

Let S be one of the rings $k[\Delta]$ or $k(\Delta)$. Thinking of S as an R-module the minimal free resolution of S has the form

$$\begin{aligned} 0 \to \bigoplus_j S[-j]^{\beta_{l,j}} \to \bigoplus_j S[-j]^{\beta_{l-1,j}} \to \cdots \\ \to \bigoplus_j S[-j]^{\beta_{1,j}} \to \bigoplus_j S[-j]^{\beta_{0,j}} \to S \to 0. \end{aligned}$$

Here $S[-j]$ is the module S shifted by degree j and l is the length of the resolution, also called the *homological dimension* or *projective dimension* of S. The $\beta_{i,j}$ are called the *(graded)*

Betti numbers of S. Using the Auslander-Buchsbaum formula [35, Theorem I.11.2], for a Cohen-Macaulay complex Δ the minimal free resolution of $k[\Delta]$ has length $n-d$ and the minimal free resolution of $k(\Delta)$ has length n (where n is the number of vertices in Δ and hence also the number of variables in R). The Betti numbers of $k[\Delta]$ are related to the topology of the complex Δ. One powerful expression of this relationship is Hochster's Formula [11],

$$\beta_{i,j}(k[\Delta]) = \sum_{W \subset V, |W|=j} \dim_k(\tilde{H}_{j-i-1}(\Delta_W; k)). \tag{2.1}$$

2.1.6 Orders on Monomials

We now turn our attention to terminology related to monomial ideals. Given a monomial $m = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$ in the indeterminates $X_1, \ldots, X_n$ define the *degree* of m to be $\sum_{i=1}^n a_i$. A non-empty set M of monomials is an *order ideal* if for all $m \in M$ and $m'|m$ we have $m' \in M$. Denote by M_i the monomials in M of degree i. The *degree sequence* of M is the vector $(|M_0|, |M_1|, |M_2|, \ldots)$. A vector that is the degree sequence of some order ideal of monomials is called an *M-vector*.

For monomials of the same degree define the *lexicographic order* $<_l$ by $X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m} <_l X_1^{b_1} X_2^{b_2} \cdots X_m^{b_m}$ if there exists some $k \in \{1, 2, \ldots, m\}$ such that $a_i = b_i$ for $i < k$ and $a_k > b_k$. Let $L \subseteq k[x_1, \ldots, x_n]$ be an ideal generated by monomials. Then L is a *lex ideal* if given any two monomials m and m' of the same degree with $m <_l m'$ and $m' \in L$ then $m \in L$. Pick any M-vector $\mathbf{h} = (h_0, h_1, \ldots, h_d)$ and let $R = k[x_1, \ldots, x_{h_1}]$. Then by Macaulay's theorem [19] there is a lex ideal $L \subseteq R$ such that $F(R/L, i) = h_i$ for all i.

Another important order on monomials is the *reverse-lexicographic order* or rev-lex order, denoted $<_{rl}$. We define the rev-lex order by $X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m} <_{rl} X_1^{b_1} X_2^{b_2} \cdots X_m^{b_m}$ if there exists some $k \in \{1, 2, \ldots, m\}$ such that $a_i = b_i$ for $i > k$ and $a_k < b_k$. An order ideal M is *compressed* if for each j the elements of M_j are the first $|M_j|$ monomials of degree j in the rev-lex order. Given any M-vector $\mathbf{h}$ there exists a compressed order ideal of monomials with degree sequence equal to $\mathbf{h}$ [4, Proposition 1].

Let m be a monomial and $c \in \mathbb{N}$ such that the degree of m is less than or equal to c. There is a unique way to write m as $m = X_{e_1}X_{e_2}\cdots X_{e_c}$ where $0 \leq e_1 \leq e_2 \leq \cdots \leq e_c$ and we take $X_0 = 1$. This is called the *extended representation* of m. Define the *partial order* $<_p$ on monomials of degree less than or equal to c by $X_{e_1}X_{e_2}\cdots X_{e_c} \leq_p X_{e'_1}X_{e'_2}\cdots X_{e'_c}$ if and only if $e_k \leq e'_k$ for $k = 1, \ldots, c$. Note that any initial segment of monomials in the rev-lex order is also an initial segment in this partial order. For $C \in \mathbb{N}$ we also define a partial order $<_p$ on C-subsets of the natural numbers. If $S = \{i_1, \ldots, i_C\}$ and $T = \{j_1, \ldots, j_C\}$ are C-subsets of $\mathbb{N}$ with elements listed in increasing order then $S \leq_p T$ if and only if $i_k \leq j_k$ for $k = 1, \ldots, C$.

2.1.7 M-vectors

In addition to the definition of an M-vector in terms of order ideals there is also a numerical characterization of M-vectors due to Macaulay [19]. Given any $l, i \in \mathbb{N}$ there is a unique expansion of l of the form

$$l = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}, \qquad n_i > n_{i-1} > \cdots > n_j \geq j \geq 1.$$

This is called the *i-canonical representation of l*. Define $l^{<i>}$, the *ith pseudo-power of l*, by

$$l^{<i>} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}.$$

An integer vector $(h_0, h_1, \ldots)$ is an M-vector if and only if $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{<i>}$ for $i \geq 1$.

Another characterization of M-vectors is in terms of Cohen-Macaulay complexes. An integer vector $\mathbf{h} = (h_0, h_1, \ldots, h_d)$ is an M-vector if and only if there exists a $(d-1)$-dimensional Cohen-Macaulay complex Δ such that $h(\Delta) = \mathbf{h}$. This result is also true for the more restrictive class of shellable complexes [35, Theorem II.3.3].

2.2 Previous Work on the h-vectors of Homology Balls

In this section we discuss some of the previously known necessary conditions on the h-vectors of homology balls as well as some conjectured conditions on these h-vectors. Many of the results are obtained by examining the relationship between the h-vector of a homology ball and the h-vector of its boundary homology sphere. We use the conditions that the g-conjecture places on the h-vector of the boundary sphere to obtain possible restrictions on the h-vector of the original ball.

Conjecture 2.2. *(The g-Conjecture) An integer vector $(h_0, h_1, \ldots, h_d)$ with $h_0 = 1$ is the h-vector of a homology d-sphere if and only if*

1. $h_i = h_{d-i}$ *for* $0 \leq i \leq \lfloor d/2 \rfloor$, *and*
2. $(1, g_1, g_2, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-vector, where* $g_i = h_i - h_{i-1}$.

   (This is the g-vector of the homology d-sphere).

The g-conjecture is not known to hold for all homology spheres (or all triangulated spheres) but has been proved for boundaries of simplicial polytopes [4, 32]. The relations of condition one of the g-conjecture are called the Dehn-Sommerville equations and are known to hold for all homology spheres [14]. Additionally, using Barnette's Lower

Bound Theorem [1] and standard algebraic arguments one can show that for all triangulated spheres the initial part (g_0, g_1, g_2) of the g-vector is an M-vector (in fact this result is true for the much larger class of all doubly Cohen-Macaulay complexes, which include all homology spheres [27]). Therefore the g-conjecture holds for all homology spheres of dimension four or lower.

In order to use these known and conjectured conditions on the h-vectors of homology spheres we need a relationship between the h-vector of a homology ball and the h-vector of its boundary. In [20], MacDonald proves a generalization of the Dehn-Sommerville equations for triangulated manifolds with possibly non-empty boundary. Here we will use the equivalent result for homology manifolds expressed in terms of h-vectors. This form of the result is due independently to Gräbe [9, Section 2.2] and Novik and Swartz [28, Theorem 3.1].

Theorem 2.3. *Let Δ be a $(d-1)$-dimensional homology manifold with boundary. Then for all $0 \le i \le d$,*

$$h_{d-i}(\Delta) - h_i(\Delta) = \binom{d}{i}(-1)^{d-1-i}\tilde{\chi}(|\Delta|) - g_i(\partial\Delta).$$

In the case where Δ is a $(d-1)$-dimensional homology ball, this reduces to $h_i(\Delta) - h_{d-i}(\Delta) = g_i(\partial\Delta)$. Let Δ_k be the cone over Δ taken k times. Then Δ_k is a $(d-1+k)$-dimensional homology ball with boundary a $(d-2+k)$-dimensional homology sphere. Following the argument of Billera and Lee [3, Corollary 3.14] yields $g_i(\partial\Delta_k) = h_i(\Delta_k) - h_{d+k-i}(\Delta_k) = h_i(\Delta) - h_{d+k-i}(\Delta)$, $0 \le i \le d+k$. Combining this result with the g-conjecture we have the following set of conditions.

Conjecture 2.4. *If $(h_0, \ldots, h_d)$ is the h-vector of a homology $(d-1)$-ball and we take $h_i = 0$ for $i > d$ then $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, \ldots, h_m - h_{d+k-m})$ is an M-vector for $k = 0, \ldots, d+1$, $m = \lfloor (d+k-1)/2 \rfloor$.*

If the g-conjecture was proved for homology spheres then Conjecture 2.4 would follow as a corollary. However, many of the conditions in Conjecture 2.4 can be verified using known results.

Proposition 2.5. *Conjecture 2.4 holds for $d-3 \leq k \leq d+1$, and for all k the vector $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, h_2 - h_{d+k-2})$ is an M-vector.*

Proof. The case $k = d+1$ is the statement that the original h-vector of the ball, $(h_0, \ldots, h_d)$, is an M-vector. Since homology balls are Cohen-Macaulay complexes their h-vectors are M-vectors. For $d-2 \leq k \leq d$, since $h_d(\Delta) = 0$ for any homology ball Δ, Conjecture 2.4 reduces to the fact that the h-vector of Δ is an M-vector, which was discussed above.

For the case $k = d-3$ we must show that $(h_0, h_1, \ldots, h_{d-3}, h_{d-2} - h_{d-1})$ is an M-vector. Since we already know that the h-vector of Δ is an M-vector, we only need to prove that $h_{d-2} - h_{d-1}$ is non-negative. As a consequence of [36, Corollary 4.29] $h_{d-2} \geq h_{d-1}$ for any homology ball, giving the desired inequality.

Finally, Barnette's result implies that the initial segments $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, h_2 - h_{d+k-2})$ of the vectors in Conjecture 2.4 are M-vectors. $\square$

As a consequence of Proposition 2.5 we have the following.

Corollary 2.6. *Conjecture 2.4 holds for homology balls of dimension less than or equal to four.*

2.3 A New Type of Necessary Condition

Recall that a polyhedral ball is a triangulated ball obtained from the boundary of a simplicial polytope by removing a single vertex (and all faces containing this vertex). In particular, all polyhedral balls are also homology balls. In [3, Conjecture 5.1], Billera and Lee made the following conjecture.

Conjecture 2.7 (Billera and Lee)**.** *The conditions of Conjecture 2.4 are necessary and sufficient to describe the h-vectors of polyhedral balls.*

For polyhedral balls necessity is a consequence of the g-theorem for polytopes [3, Corollary 3.14]. Sufficiency was verified for three-balls by Lee and Schmidt [17]. In dimension four, Lee and Schmidt obtained all of the conjectured h-vectors using shellable (but not necessarily polyhedral) balls. In this section we show that in dimensions higher than four there are certain vectors that satisfy the conditions of Conjecture 2.4 but are not the h-vectors of any homology ball. This provides counterexamples to sufficiency in Conjecture 2.7 in dimensions five and higher.

For the results of this section we need the idea of splitting a simplicial complex along a codimension-one face. Let Δ be a $(d-1)$-homology ball and F be an interior $(d-2)$-face of Δ. If $\Delta_{[n]\setminus F}$ is not connected, let W_1 and W_2 be the vertex sets of the two components. Define the *splitting* of Δ along F to be the creation of two new simplicial complexes $\Delta_{W_1 \cup F}$ and $\Delta_{W_2 \cup F}$. Looking at Mayer-Vietoris sequences shows that these two new complexes are also homology balls.

2.3.1 The Betti Diagram of the Face Ring

The goal of this section is to develop a method for proving that for a certain class of h-vectors, every homology ball with one of these h-vectors must admit a splitting along some codimension-one face. We first study the upper right entry of the Betti diagram of the face ring modulo a linear system of parameters, $\beta_{n-d,n-d+1}(k[\Delta])$, and its relationship to the existence of a splitting of Δ along a codimension-one face.

Proposition 2.8. *Let Δ be a $(d-1)$-homology ball. Then $\beta_{n-d,n-d+1}(k[\Delta]) > 0$ if and only if there is a splitting of Δ along a codimension-one face that creates two homology balls.*

Proof. From Hochster's Formula (2.1)

$$\beta_{n-d,n-d+1}(k[\Delta]) = \sum_{W \subset [n], |W| = n-d+1} \dim_k(\tilde{H}_0\,(\Delta_W; k)). \tag{2.2}$$

From [10, Lemma 3.7], the group $\tilde{H}_0\,(\Delta_W; k)$ in equation (2.2) is trivial whenever $[n]\backslash W$ is not a face of Δ. Therefore, $\beta_{n-d,n-d+1}(k[\Delta])$ is non-zero if and only if there exists a $(d-2)$-face F of Δ such that $\Delta_{[n]\backslash F}$ is not connected. Hence $\beta_{n-d,n-d+1}(k[\Delta])$ is non-zero if and only if Δ can be split along a codimension-one face to obtain two homology balls. $\square$

Proposition 2.8 gives us a nice way to test if a particular ball can be split. However, we want a condition that shows that all balls with a given h-vector are splittable. Given a homology ball Δ, let $R = k[x_1, \ldots, x_n]$ as in the definition of the face ring. Let L be the lexicographic ideal such that the Hilbert function of R/L is equal to the h-vector of Δ. Since Δ is a homology ball, it's h-vector is an M-vector, so such an ideal L exists (see Section 2.1.6).

In the next Proposition we use Irena Peeva's technique of consecutive cancellations to relate the Betti numbers of R/L and $k[\Delta]$. This allows us to detect the presence of a face along which we can split Δ by looking at the Betti diagram of R/L. Since R/L depends only on the h-vector of Δ, this result allows us to show that all balls with certain h-vectors can be split.

Proposition 2.9. *Let Δ be a $(d-1)$-homology ball. Let L be the lexicographic ideal such that the Hilbert function of R/L is equal to the h-vector of Δ. If $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L) > 0$ then there is a splitting of Δ along a codimension-one face that creates two homology balls.*

Proof. We will prove that whenever $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L) > 0$ we must have $\beta_{n-d,n-d+1}(k[\Delta]) > 0$. The result then follows by Proposition 2.8. By [35, Theorem I.12.4], $\beta_{n-d,n-d+1}(k[\Delta])$ is equal to the dimension of the degree one portion of the socle of $k(\Delta)$. Since $k(\Delta)$ has projective dimension n, the dimension of the degree i portion of the socle of $k(\Delta)$ is given by the Betti number $\beta_{n,n+i}(k(\Delta))$ [35, Theorem I.12.4]. Combining these facts gives

$$\beta_{n-d,n-d+1}(k[\Delta]) = \dim_k(\operatorname{soc} k(\Delta))_1 = \beta_{n,n+1}(k(\Delta)).$$

Since the Hilbert Function of $k(\Delta)$ is equal to the h-vector of Δ, we can use Peeva's cancellation technique to relate the h-vector of Δ to the Betti numbers of $k(\Delta)$. By [29, Theorem 1.1], the Betti number $\beta_{n,n+1}(k(\Delta))$ is bounded above by $\beta_{n,n+1}(R/L)$ and below by $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L)$. Therefore, if $\beta_{n,n+1}(R/L) - \beta_{n-1,n+1}(R/L) > 0$ we have $\beta_{n-d,n-d+1}(k[\Delta]) = \beta_{n,n+1}(k(\Delta)) > 0$, which implies the desired result. □

2.3.2 Splitting Balls

Next we investigate the effect on the h-vector when a homology ball is split along a single codimension-one face.

Proposition 2.10. *Let Δ_1 and Δ_2 be two homology $(d-1)$-balls that can be joined along a common homology $(d-2)$-ball B to form a single homology $(d-1)$-ball Δ. Then*

$$h_i(\Delta_1) + h_i(\Delta_2) = h_i(\Delta) + (h_i(B) - h_{i-1}(B)) \tag{2.3}$$

where we take $h_{-1}(B) = h_d(B) = 0$.

Proof. On the level of f-vectors

$$f_i(\Delta_1) + f_i(\Delta_2) = f_i(\Delta) + f_i(B).$$

The result then follows from a straightforward calculation using the definition of the h-vector. $\square$

As a special case of equation (2.3), when we join two homology balls along a single codimension-one face the h-vector of the resulting complex is the sum of the h-vectors of the two component homology balls but with h_0 reduced by one (causing h_0 of the resulting complex to still equal one) and h_1 increased by one.

Example 2.11 (Algebraic approach applied to (1,4,5,7,3,2,0)). With these tools we now consider the h-vector $(1, 4, 5, 7, 3, 2, 0)$. Assume there is a homology five-ball Δ with this h-vector. Using the result of Eliahou and Kervaire [8, Section 3], we calculate the Betti numbers of R/L, where L is the lexicographic ideal such that the Hilbert function of R/L is equal to $(1, 4, 5, 7, 3, 2, 0)$. This yields $\beta_{n,n+1}(R/L) = 1$ and $\beta_{n-1,n+1}(R/L) = 0$. Therefore, by Proposition 2.9 there exists a codimension-one face along which we can split Δ.

Using formula (2.3) for the h-vector obtained by combining homology balls, we now look for possible h-vectors for the two homology five-balls created when we split our original homology ball. Each of the two smaller homology balls must satisfy all of the known portions of Conjecture 2.4 as discussed in Proposition 2.5.

The only two pairs of options for the h_1-values of our smaller homology balls are 2,1 or 3,0. If we take 2 and 1 as the h_1-values the largest possible corresponding values of h_2 are 3 and 1, which do not sum to the h_2-value of our original ball. If we take 3 and 0 as our h_2-values then one homology ball must have h-vector $(1, 3, 5, 7, 3, 2, 0)$. However, the

g-vector of the boundary homology four-sphere of this ball would be (1,1,2), violating a known portion of the g-conjecture.

We have shown that there must be a division of our homology five-ball into two smaller homology balls, but also that no such division exists. Therefore $(1, 4, 5, 7, 3, 2, 0)$ is not the h-vector of a homology five-ball, even though it satisfies all of the conditions of Conjecture 2.4.

Similar calculations show that there are other h-vectors that satisfy the conditions of Conjecture 2.4 but are not the h-vectors of homology balls. Examples include $(1, 4, 6, 9, 4, 2, 0)$ and $(1, 5, 6, 8, 4, 3, 0)$.

2.3.3 A Combinatorial Approach

Some of the results of the previous section can also be obtained using a more combinatorial viewpoint. Given an h-vector $(1, h_1, \ldots, h_d)$ and the corresponding f-vector $(1, f_0, f_1, \ldots, f_{d-1})$, look at all of the possible graphs with f-vector $(1, f_0, f_1)$. For each of these graphs count the maximum number of triangles possible in a simplicial complex of dimension $d-1$ with the given graph as its one-skeleton (or equivalently, the number of triangles in the $(d-1)$-skeleton of the flag complex or clique complex induced by the graph). Compute h'_3 where $(1, h_1, h_2, h'_3, h'_4, \ldots, h'_d)$ is the h-vector of a $(d-1)$-complex with all possible triangles. Since removing triangles from a complex Δ decreases $h_3(\Delta)$ and adding faces of dimension greater than two does not change $h_3(\Delta)$, any one-skeleton of a complex with h-vector $(1, h_1, \ldots, h_d)$ must have $h_3 \leq h'_3$.

This approach gives us a collection containing all possible one-skeletons for balls with h-vector $(1, h_1, \ldots, h_d)$. In some cases, we are able to use other facts we know about homology balls to show that there cannot be any balls with the given h-vector and any of these one-skeletons.

Example 2.12 (Combinatorial approach applied to (1,4,5,7,3,2,0)). Doing an exhaustive search we find that for the h-vector $(1,4,5,7,3,2,0)$ the only graphs that obtain $h'_3 \geq h_3$ have a vertex of degree less than or equal to five. In a homology ball each vertex must be contained in at least one facet. This forces each vertex in a homology five-ball to have degree at least five. If a vertex has degree exactly five then it is contained in only one facet. In this case we can remove the facet to create a homology five-ball with h-vector $(1,3,5,7,3,2,0)$. As argued in the previous section, this contradicts the known conditions of Conjecture 2.4. Therefore each vertex must have degree at least six, so no homology five-ball with h-vector $(1,4,5,7,3,2,0)$ exists.

Using this approach, we next describe an infinite collection of h-vectors that satisfy all of the conditions of Conjecture 2.4 but such that the existence of a homology ball with one of these h-vectors would contradict the known fact that $g_2 \leq g_1^{<1>}$ for a sphere. This shows that the conditions of Conjecture 2.4 are not sufficient in dimensions five and higher.

Theorem 2.13. *If x, y are integers with $x > 4$ and $1 < y < x$ then*

$$\begin{aligned}\Bigg(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \binom{x+1}{3} - 2, \ldots,\\ \binom{x+1}{3} - 2, \binom{x}{2} - \left(\binom{y}{2} + 1\right), x - y, 0\Bigg)\end{aligned} \tag{2.4}$$

satisfies the conditions in Conjecture 2.4 but is not the h-vector of a homology ball.

Proof. We first show that the vector (2.4) satisfies all of the conditions in Conjecture 2.4.

The case $k = 0$ (the boundary sphere condition) requires that

$$\left(1, y, \binom{y}{2} + 1\right)$$

is an M-vector, which follows since $\binom{y}{2} < \binom{y+1}{2}$.

For the case $k=1$ (the condition that comes from taking a cone)

$$\left(1, x, \binom{x}{2} - x + y, \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2} - 1\right)$$

must be an M-vector. To see this, first note that $\binom{x}{2} - x + y = \binom{x}{2} - \binom{x-1}{1} + y - 1 = \binom{x-1}{2} + \binom{y-1}{1}$. Therefore the corresponding second pseudopower is $\left(\binom{x}{2} - x + y\right)^{<2>} = \binom{x}{3} + \binom{y}{2} = \binom{x}{3} + \binom{x}{2} - \binom{x}{2} + \binom{y}{2} = \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2}$. Combining this with the fact that $x > y$ shows that the desired vector is an M-vector.

For the case $k=2$

$$\left(1, x, \binom{x}{2}, \binom{x+1}{3} - 2 - x + y, \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2} - 1\right)$$

must be an M-vector. Since $x > y$, the step from the second to third entry satisfies Macaulay's condition. Note that since $x > y > 1$ and $x > 4$, $\binom{x}{2} - \binom{y}{2} = \frac{1}{2}(x(x-1) - y(y-1)) > \frac{1}{2}(x(x-1) - y(x-1)) > x - y$. Therefore, the step from the third to fourth entry is non-increasing. All of the remaining checks of Macaulay's conditions and non-negativity needed to show that the desired vector is an M-vector are straightforward.

All of the higher k values result in vectors of one of the forms

$$\begin{aligned}
&\left(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \ldots,\right.\\
&\qquad \left.\binom{x+1}{3} - 2, \binom{x+1}{3} - 2 - x + y, \binom{x+1}{3} - \binom{x}{2} + \binom{y}{2} - 1\right),\\
&\left(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \ldots,\right.\\
&\qquad \left.\binom{x+1}{3} - 2, \binom{x+1}{3} - 2 - x + y, \binom{x}{2} - \binom{y}{2} - 1\right),\\
&\left(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \ldots, \binom{x+1}{3} - 2, \binom{x}{2} - x - \left(\binom{y}{2} - y\right) - 1\right),
\end{aligned}$$

or the original h-vector itself. Using the same arguments as in the previous cases, these are also all M-vectors.

Assume that there exists a homology $(d-1)$-ball Δ with h-vector given by (2.4). We will show that this results in a contradiction. Calculating the f-vector of Δ yields

$$f_0 = d + x, \qquad \binom{d+x}{2} - f_1 = x, \qquad \binom{d+x}{3} - f_2 = \frac{x^2}{2} + \left(\frac{2d-3}{2}\right) x + 2.$$

Note that $\binom{d+x}{i+1} - f_i$ counts the number of non-i-faces of Δ.

We claim that every vertex of Δ has degree at least d. To see this, first note that any vertex of degree less than $d-1$ would not be contained in any facet of our complex. If there was a vertex of degree $d-1$ this vertex would be contained in exactly one facet. Removing this facet from our homology ball would decrease h_1 by one, leaving us with a homology ball whose boundary homology sphere would have g-vector $(1, y-1, \binom{y}{2}+1, \ldots)$, contradicting the fact that for a sphere (g_0, g_1, g_2) is an M-vector.

Let G be the graph of non-edges of Δ. The vertex set of G is $[d+x]$, the same as the vertex set of Δ, and $\{a, b\}$ is an edge of G if and only if $\{a, b\} \notin \Delta$. By the above claim the maximum degree of any vertex in G is $x - 1$.

For each edge $\{a, b\}$ of G and each vertex $c \notin \{a, b\}$ the triangle $\{a, b, c\}$ is a non-triangle of Δ. If G has no vertex of degree at least two, then for each combination of a non-edge and a vertex not in that edge there is a distinct non-triangle. This results in a total of at least $x(x+d-2)$ non-triangles, far more than the $\frac{x^2}{2} + \left(\frac{2d-3}{2}\right) x + 2$ allowed non-triangles. We can therefore assume that G has a vertex v of degree k where $2 \leq k < x$.

Label the edges of G by $e_1, e_2, \ldots, e_x$ where v is contained in e_i for $1 \leq i \leq k$. Let G_i, $1 \leq i \leq x$, be the graph on the vertex set $[x+d]$ with edges $\{e_j\}_{j=1}^{i}$. Let $A_i := \{\{a, b, c\} \subset$

$[x+d]: a,b,c$ distinct and G_i contains at least one of the edges $\{a,b\},\{b,c\},\{a,c\}\}$. Then $|A_x|$ is less than or equal to the number of non-triangles in Δ.

We now compare the sets A_{i-1} and A_i for $i = 1,\dots,x$. First note that $A_{i-1} \subseteq A_i$. When moving from A_{i-1} to A_i the new elements are those containing the two endpoints of e_i and one other vertex which is not adjacent in G_{i-1} to either endpoint of e_i. For $i \leq k$ this implies $|A_i \backslash A_{i-1}| = x+d-1-i$. For $i \geq k+1$ the graph G_{i-1} has $i-(k+1)$ edges that do not contain v. Therefore there are at least $(x+d-3-(i-(k+1))$ vertices that are not in e_i and are not adjacent in G_{i-1} to either endpoint of e_i. Thus $A_i \backslash A_{i-1}$ contains at least $(x+d-3-(i-(k+1))$ elements. In total, $|A_x|$ is bounded below by

$$\begin{aligned}
&(x+d-2)+(x+d-3)+\cdots+(x+d-1-k)+(x+d-3)+\cdots \\
&\qquad +(x+d-3-(x-(k+1)) \\
&= \Big((x+d-2)+(x+d-3)+\cdots+(d-1)\Big)+(k-1)(x-k) \\
&= \frac{x^2}{2}+\frac{2d-3}{2}\cdot x+(k-1)(x-k).
\end{aligned}$$

Since $x > 4$ and $k > 1$, $(k-1)(x-k) \geq 3$. Therefore $|A_x| > \binom{d+x}{3} - f_2$ which means there is at least one too many non-triangles and no homology ball with this h-vector exists. $\square$

Corollary 2.14. *The conditions of Conjecture 2.4 are not sufficient to characterize homology balls in dimensions five and higher. In particular, Conjecture 2.7 is false in all dimensions greater than four.*

2.4 Construction Methods

In this section we present a method for constructing balls with a large variety of different h-vectors. The main theorem of the section is the following. (We address the case where the dimension $d-1$ is even first; the small alterations needed in the case where $d-1$ is odd are discussed in Theorem 2.18 at the end of the section.)

Theorem 2.15. *Let $d-1$ be even and let $(1, h_1, h_2, \ldots, h_{d-1}, 0)$ satisfy the following conditions:*

- $(1, h_1 - 1, h_2 - h_1, \ldots, h_{(d-3)/2} - h_{(d-5)/2}, \max\{h_{(d-1)/2} - h_{(d-3)/2}, 0\})$ *is an M-vector.*

- $(1, h_1 - h_{d-1}, h_2 - h_{d-2}, \ldots, h_{(d-1)/2} - h_{(d+1)/2})$ *is an M-vector.*

- $h_{(d+1)/2} \geq h_{(d+3)/2} \geq \cdots \geq h_{d-1}$.

Then there is a triangulated $(d-1)$-ball with h-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.

Theorem 2.15 does not obtain all possible h-vectors of balls. In [17, Theorem 2], Lee and Schmidt show that h-vectors with $h_1 \geq h_2 \geq \ldots \geq h_{d-1} \geq h_d = 0$ are the h-vectors of triangulated balls. However, in dimensions five and higher, taking $h_i > h_{i-1}$ for $2 \leq i \leq (d/2 - 1)$ violates the conditions of Theorem 2.15 (or Theorem 2.18 for $d-1$ odd). Additionally, the construction of Billera and Lee in [4, Section 6] can be used to create balls with h-vector equal to any M-vector whose second half is all zeros, many of which cannot be obtained using our construction.

In the proof of Theorem 2.15 we divide a sphere into two complementary balls that intersect only along their common boundary. The following lemma describes the relationship between the h-vectors of the two balls and the original sphere in this situation.

Lemma 2.16. *Let Δ be a $(d-1)$-dimensional triangulated sphere and $B \subset \Delta$ be a $(d-1)$-dimensional triangulated ball. Let $C := (\Delta \backslash B) \cup (\partial B)$ be the complementary $(d-1)$-ball to B in Δ. Then $h_i(C) = h_i(\Delta) - h_{d-i}(B)$.*

Proof. Since B and C intersect only along ∂B and $B \cup C = \Delta$

$$f_i(B) + f_i(C) = f_i(\Delta) + f_i(\partial B).$$

A straightforward calculation shows

$$h_i(B) + h_i(C) = h_i(\Delta) + (h_i(\partial B) - h_{i-1}(\partial B)) = h_i(\Delta) + g_i(\partial B), \tag{2.5}$$

where we take $g_0(\partial B) = h_0(\partial B) = 1$. From Theorem 2.3, $g_i(\partial B) = h_i(B) - h_{d-i}(B)$. Substituting this into equation (2.5) and simplifying yields $h_i(C) = h_i(\Delta) - h_{d-i}(B)$, as desired. □

Since the proof of Theorem 2.15 relies heavily on the ideas in the Billera-Lee construction [4, Section 6] and Kalai's paper [13] we review some of those concepts and notation here.

Let $d > 0$ be an odd integer. Define $F_d(n)$, a collection of $(d+1)$-subsets of $[n]$, by $F \in F_d(n)$ if and only if $F = \cup_{j=1}^{(d+1)/2}\{i_j, i_j + 1\}$ where $i_1, \ldots, i_{(d+1)/2}$ are elements of $[n-1]$ such that $i_{j+1} > i_j + 1$ for every j. Let J be any initial segment in $F_d(n)$ with respect to the partial order $<_p$. Kalai showed that $B(J)$, the simplicial complex with facets the elements of J, is a shellable ball. A shelling order is given by any linear order of the facets consistent with $<_p$ and the size of the restriction face of a facet F is given by the number of pairs of vertices of F not in their leftmost possible position, $|r(F)| = |\{j : i_j \neq 2j-1\}|$.

Let $\mathcal{F}_{b,c}$ be the set of all monomials in the variables $Y_1, Y_2, \ldots, Y_b$ of degree at most c. Define a bijection $\alpha : F_d(n) \to \mathcal{F}_{n-d-1,(d+1)/2}$ by $\alpha(F) = \prod_{j=1}^{(d+1)/2} Y_{e_j}$, where $e_j = i_j - 2j + 1$ is the amount that the jth pair of F is displaced from its leftmost possible position and we take $Y_0 = 1$. This bijection is order preserving between the partial orders $<_p$ on monomials and subsets of $[n]$. Therefore, given an initial segment I of monomials in $\mathcal{F}_{b,c}$ using the partial order $<_p$, the corresponding facets (under the map α^{-1}) form a shellable ball $B(\alpha^{-1}(I))$

with $|r(F)|$ equal to the degree of $\alpha(F)$ for all facets F. For ease of notation, when I is an initial segment of monomials in the $<_p$ order we write $B(\alpha^{-1}(I))$ as $B(I)$.

In this way, any rev-lex initial segment of monomials I gives rise to a shellable ball $B(I)$. So given an M-vector, the image under α^{-1} of the corresponding compressed order ideal of monomials is the set of facets of a shellable ball with h-vector equal to the original M-vector (this is what was done in the Billera and Lee paper).

In the case where d is even, the construction is altered by adding an additional vertex $\{0\}$ to each facet.

Proof of Theorem 2.15. Define

$$(1, g_1, g_2, \ldots, g_{(d-3)/2}, g_{(d-1)/2}) :=$$
$$(1, h_1 - 1, h_2 - h_1, \ldots, h_{(d-3)/2} - h_{(d-5)/2}, \max\{h_{(d-1)/2} - h_{(d-3)/2}, 0\}).$$

First consider the case $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$. Since $(1, g_1, g_2, \ldots, g_{(d-1)/2}, 0, 0, \ldots)$ is an M-vector, there is a compressed order ideal I with this vector as its degree sequence. We know $I \subseteq \mathcal{F}_{g_1,(d-1)/2} \subseteq \mathcal{F}_{g_1,(d+1)/2}$, so using the Billera-Lee method we construct the d-ball $B(I)$ with h-vector $(1, g_1, g_2, \ldots, g_{(d-1)/2}, 0, 0, \ldots, 0)$. Note that each facet of $B(I)$ contains the elements 1 and 2. The boundary of $B(I)$ is a $(d-1)$-sphere. Theorem 2.3, the definition of the g_i, and the Dehn-Sommerville equations combine to give

$$h(\partial B(I)) = (1, h_1, h_2, \ldots, h_{(d-1)/2}, h_{(d-1)/2}, \ldots, h_2, h_1, 1).$$

Define

$$(1, G_1, \ldots, G_{(d-1)/2}) := (1, h_1 - h_{d-1}, \ldots, h_{(d-1)/2} - h_{(d+1)/2}).$$

We now construct a $(d-1)$-ball $\mathcal{B}$ such that $\mathcal{B}$ is contained in the sphere $\partial B(I)$ and $h(\mathcal{B}) = (1, G_1, G_2, \ldots, G_{(d-1)/2)}, 0, 0, \ldots, 0)$. Using Lemma 2.16 we have that the complementary ball $\partial B(I) \backslash (\mathcal{B} \backslash \partial \mathcal{B})$ is the desired $(d-1)$-ball.

The $(d-1)$-ball $\mathcal{B}$ uses the same correspondence α between facets and monomials except that the vertex names are shifted by one. Given a monomial $m = \prod_{j=1}^{(d-1)/2} Y_{e_j}$ we define the potential corresponding facet of $\mathcal{B}$ by $(\alpha')^{-1}(m) := \{1\} \cup \left(\cup_{j=1}^{(d-1)/2} \{i_j, i_j + 1\} \right)$ where $i_j = e_j + 2j$ (instead of $i_j = e_j + 2j - 1$, as is the case for the correspondence α^{-1}).

Next we characterize the facets of $\partial B(I)$ that will be used in the construction of $\mathcal{B}$. A set of d vertices is a facet of $\partial B(I)$ if and only if it is in exactly one facet of $B(I)$. Note that the only possible facet of $B(I)$ that can contain the face $(\alpha')^{-1}(\prod_{j=1}^{(d-1)/2} Y_{e_j})$ is $\alpha^{-1}(Y_0 \cdot \prod_{j=1}^{(d-1)/2} Y_{e'_j})$ where $e'_j = \max\{e_j - 1, 0\}$. It follows that $(\alpha')^{-1}(\prod_{j=1}^{(d-1)/2} Y_{e_j})$ is a facet of $\partial B(I)$ if and only if $\prod_{j=1}^{(d-1)/2} Y_{e'_j} \in I$. Additionally, since all of the facets of $B(I)$ contain the element 1, the face $F \backslash \{1\}$ is in $\partial B(I)$ for all facets F of $B(I)$.

We now build the ball $\mathcal{B}$. For each k we will select a set $\mathcal{M}_k$ of degree k monomials such that $|\mathcal{M}_k| = G_k$. We will show that $\cup_{i=0}^{(d-1)/2} \mathcal{M}_i$ is an initial segment in the partial order $<_p$. Then the facets of $\mathcal{B}$ will be the faces $(\alpha')^{-1}(m)$ for $m \in \cup_{i=0}^{(d-1)/2} \mathcal{M}_i$.

By the above discussion, since $1 \in I$ we know that $(\alpha')^{-1}(1)$ is a facet of $\partial B(I)$. We therefore set $\mathcal{M}_0 = \{1\}$.

Assume that for some $k > 0$ we have already chosen $\mathcal{M}_i$ for $i \leq k$ with $|\mathcal{M}_i| = G_i$. Define the set S_{k+1} to be $\{Y_1 \cdot m : m \in \mathcal{M}_k\}$ (call these type one elements) as well as all of the monomials $\prod_{i=1}^{k+1} Y_{e_i}$ such that all of the $e_i > 1$ and $\prod_{i=1}^{k+1} Y_{e_i - 1} \in I$ (these are called type two elements). There are G_k elements of the first type and g_{k+1} elements of the second type giving a total of

$$G_k + g_{k+1} = (h_k - h_{d-k}) + (h_{k+1} - h_k) = h_{k+1} - h_{d-k} \geq h_{k+1} - h_{d-(k+1)} = G_{k+1}$$

elements in S_{k+1}.

Select the first G_{k+1} elements of S_{k+1} in the rev-lex order to be the monomials in $\mathcal{M}_{k+1}$. We complete the proof of the case $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$ with the following proposition.

Proposition 2.17. *For $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$, $\cup_{i=0}^{(d-1)/2} \mathcal{M}_i$ is an initial segment in the order $<_p$.*

We prove Proposition 2.17 inductively; given that the monomials in $\cup_{i=0}^{k} \mathcal{M}_i$ are an initial segment in $<_p$ we show that the monomials in $\cup_{i=0}^{k+1} \mathcal{M}_i$ still form an initial segment. The base case $k = 0$ follows since $\mathcal{M}_0 = \{1\}$ is an initial segment in $<_p$. We divide the proof of the inductive step into five claims.

Claim 1. *Let τ be a type two element in $\mathcal{M}_k$. Then $\{m \in \mathcal{M}_k : m \leq_{rl} \tau\}$ is a rev-lex initial segment in degree k.*

Proof of Claim 1. Let m be a degree k monomial such that $m <_{rl} \tau$. Let m' be the monomial m with all of the Y_1's changed to Y_2's (if m does not contain Y_1 then $m' = m$). Both m and τ are degree k monomials and τ contains no Y_1's which implies $m' \leq_{rl} \tau$. Since I is a compressed order ideal the type two elements of S_k form a compressed order ideal in the variables $Y_2, Y_3, \ldots$. Therefore m' must be a type two element of S_k and $m' \in \mathcal{M}_k$. However, $m <_p m'$ which by the inductive assumption implies $m \in \mathcal{M}_k$, proving the desired claim. □

Claim 2. *Let $i \leq k$. If there exists a type two element $\tau \in S_i$ such that $\tau \notin \mathcal{M}_i$ then $\mathcal{M}_i$ is a rev-lex initial segment in degree i.*

Proof of Claim 2. Our proof is by induction on i. The result is trivial in the base case $i = 1$ where all initial segments in the order $<_p$ are also rev-lex initial segments.

Assume the claim holds for $i = l - 1 \geq 1$. Let M be the rev-lex largest element of $\mathcal{M}_l$. If M is a type two element then we are done by Claim 1, so assume that M is a type one element. Let m be a degree l monomial such that $m <_{rl} M$. We must show that $m \in \mathcal{M}_l$.

If m does not contain the variable Y_1, then the fact that $m <_{rl} \tau$ means that m is a type two element of S_l. Then the fact that $m <_{rl} M$ forces $m \in \mathcal{M}_l$.

If m contains Y_1 then $m/Y_1 <_{rl} M/Y_1 \leq_{rl} \tau/Y_j$ where Y_j is the smallest (positive) index such that Y_j is in τ. We also know that $M/Y_1 \in \mathcal{M}_{l-1}$ and τ/Y_j is a type two element in S_{l-1}. If $\tau/Y_j \in \mathcal{M}_{l-1}$ then by Claim 1 we have $m/Y_1 \in \mathcal{M}_{l-1}$. This means m is a type one element of S_l which forces $m \in \mathcal{M}_l$. If $\tau/Y_j \notin \mathcal{M}_{l-1}$ then by the inductive hypothesis and the fact that $M/Y_1 \in \mathcal{M}_{l-1}$ we have $m/Y_1 \in \mathcal{M}_{l-1}$ which forces $m \in \mathcal{M}_l$, completing the proof of the claim. □

Claim 3. *Let ρ be a type two element in $\mathcal{M}_{k+1}$. Then $\{m \in \mathcal{M}_{k+1} : m \leq_{rl} \rho\}$ is a rev-lex initial segment in degree $k+1$.*

Proof of Claim 3. Our proof of Claim 3 is in two cases. First consider the case where there exists a type two element $\tau \in S_k$ such that $\tau \notin \mathcal{M}_k$. By Claim 2 the set $\mathcal{M}_k$ is a rev-lex initial segment in degree k. Let N be the rev-lex smallest degree $k+1$ monomial not in $\mathcal{M}_{k+1}$. It is sufficient to show $\rho <_{rl} N$.

If N is not one of the first G_{k+1} monomials of degree $k+1$ in the rev-lex order then all of the elements of $\mathcal{M}_{k+1}$ are rev-lex less than N, proving the desired claim. We therefore assume that N is one of the first G_{k+1} monomials of degree $k+1$ in the rev-lex order. If N contains the variable Y_1 then since $\mathcal{M}_k$ is a rev-lex initial segment and the G_i form an M-vector we know $N/Y_1 \in \mathcal{M}_k$. This means N is a type one element in S_{k+1} and therefore $N \in \mathcal{M}_{k+1}$, contradicting our definition of N. Thus N does not contain the variable Y_1.

Since the type two elements of $\mathcal{M}_{k+1}$ are a rev-lex initial segment in $Y_2, Y_3, \ldots$ all of the type two elements of $\mathcal{M}_{k+1}$ are rev-lex less than N, proving the desired claim for the first case.

Now consider the case where all of the type two elements of S_k are in $\mathcal{M}_k$. Let n be a degree $k+1$ monomial with $n <_{rl} \rho$. We need to show that $n \in \mathcal{M}_{k+1}$. If n does not contain Y_1 then the result follows from the initial segment property of the type two elements of S_{k+1}. If n contains Y_1, then consider $n/Y_1 \leq_{rl} \rho/Y_j$ where Y_j is the smallest variable in ρ. Since the g_k form an M-vector and all of the type two elements of degree k are in $\mathcal{M}_k$ we know ρ/Y_j is a type two element in $\mathcal{M}_k$. Claim 1 then shows that n/Y_1 is in $\mathcal{M}_k$ which forces $n \in S_{k+1}$ and $n \in \mathcal{M}_{k+1}$. This completes the proof of Claim 3. □

Claim 4. *$\mathcal{M}_{k+1}$ is an initial segment of degree $k+1$ monomials in the partial order $<_p$.*

Proof of Claim 4. To prove Claim 4 we take any monomial $m \in \mathcal{M}_{k+1}$ and any degree $k+1$ monomial m' with $m' <_p m$ and show that $m' \in \mathcal{M}_{k+1}$. Claim 3 implies that $\mathcal{M}_{k+1}$ consists of a rev-lex initial segment of degree $k+1$ monomials along with a (possibly empty) collection of additional type one monomials added in rev-lex order. Since any rev-lex initial segment is also an initial segment in the partial order, we need only consider the case where m is one of the type one monomials that is not part of the rev-lex initial segment in $\mathcal{M}_{k+1}$. Because m is a type one element m contains the variable Y_1. The fact that $m' <_p m$ forces m' to contain Y_1. Then $m'/Y_1 <_p m/Y_1$ and $m/Y_1 \in \mathcal{M}_k$ which combined with the inductive hypothesis implies $m'/Y_1 \in \mathcal{M}_k$. This means m' is a type one element of S_{k+1}. Since $m' <_{rl} m$ we know $m' \in \mathcal{M}_{k+1}$, proving Claim 4. □

Claim 5. *Let $m \in \mathcal{M}_{k+1}$ and let m' be a monomial of degree less than or equal to k with $m' <_p m$. Then $m' \in \cup_{i=0}^{k} \mathcal{M}_i$.*

Proof of Claim 5. Let Y_j be the smallest variable in m, so that $m' <_p m/Y_j$. By the inductive hypothesis it is sufficient to show $m/Y_j \in \mathcal{M}_k$.

If m is a type one element $Y_j = Y_1$ and the claim follows from the definition of a type one element. If m is a type two element we consider two cases. For the case where there exists a type two element $\tau \in S_k$ such that $\tau \notin \mathcal{M}_k$ the result follows from Claim 2 and Claim 3 along with the fact that the G_i form an M-vector. In the case where all of the type two elements of S_k are in $\mathcal{M}_k$ the result was already shown in the proof of Claim 3. □

Combining Claim 4 and Claim 5 we have that the monomials in $\cup_{i=0}^{k+1}\mathcal{M}_i$ form an initial segment in the order $<_p$. This completes the inductive step, finishing the proof of Proposition 2.17.

We now address the case where $h_{(d-1)/2} - h_{(d-3)/2} < 0$. In this case $g_{(d-1)/2} = 0$ which implies

$$h(\partial B(I)) = (1, h_1, h_2, \ldots, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-3)/2}, \ldots, h_2, h_1, 1).$$

Therefore, we alter our definition of the G_i,

$$\begin{aligned}&(1, G_1, \ldots, G_{(d-3)/2}, G_{(d-1)/2}, G_{(d+1)/2}) := \\ &\quad (1, h_1 - h_{d-1}, \ldots, h_{(d-3)/2} - h_{(d+3)/2}, h_{(d-3)/2} - h_{(d+1)/2}, h_{(d-3)/2} - h_{(d-1)/2}).\end{aligned}$$

We use the same argument as above to construct the $\mathcal{M}_k$ with $k < (d-1)/2$. Note that in this case the decreasing assumption on the h_i's implies that $G_{(d-3)/2} \geq G_{(d-1)/2} \geq G_{(d+1)/2}$. This allows us to choose $\mathcal{M}_{(d-1)/2}$ to be the first $G_{(d-1)/2}$ type one elements of degree $(d-1)/2$ in the rev-lex order. Then $\cup_{k=0}^{(d-1)/2}\mathcal{M}_k$ is an initial segment in $<_p$, so the corresponding facets form a shellable ball with h-vector $(1, G_1, \ldots, G_{(d-3)/2}, G_{(d-1)/2}, 0, 0, \ldots)$.

Let E be the first $G_{(d+1)/2}$ monomials in $\mathcal{M}_{(d-1)/2}$ using the rev-lex order and let $m \in E$. The degree of m is $(d-1)/2$ meaning that all pairs of vertices in $(\alpha')^{-1}(m)$ are shifted to the right and $2 \notin (\alpha')^{-1}(m)$. However, 2 is an element of every facet of $B(I)$ and since $m \in \mathcal{M}_{(d-1)/2}$ we know $(\alpha')^{-1}(m)$ is contained in exactly one facet of $B(I)$. Therefore $(\alpha')^{-1}(m) \cup \{2\}$ must be a facet of $B(I)$. As argued above this implies that $\gamma(m) := (\alpha')^{-1}(m) \cup \{2\} \backslash \{1\}$ is in $\partial B(I)$ for all $m \in E$.

In order to get the desired $(d+1)/2$ entry of our h-vector, for each $m \in E$ we add the facet $\gamma(m)$ to our complementary ball $\mathcal{B}$. Though the facets in this last step do not correspond to monomials using the map α', because of the relationship between $(\alpha')^{-1}(m)$ and $\gamma(m)$ it is straightforward to check that adding the $\gamma(m)$ to the end of the shelling in the same order as the $(\alpha')^{-1}(m)$ still gives a shellable ball with the correct h-vector. $\square$

For the case where $d-1$ is odd we can prove the follwing result using the same argument as the above proof with appropriate changes in notation and parity.

Theorem 2.18. *Let $d-1$ be odd and let $(1, h_1, h_2, \ldots, h_{d-1}, 0)$ satisfy the following conditions:*

- $(1, h_1 - 1, h_2 - h_1, \ldots, h_{d/2-1} - h_{d/2-2}, \max\{h_{d/2} - h_{d/2-1}, 0\})$ *is an M-vector.*

- $(1, h_1 - h_{d-1}, h_2 - h_{d-2}, \ldots, h_{d/2-1} - h_{d/2+1})$ *is an M-vector.*

- $h_{d/2} \geq h_{d/2+1} \geq \cdots \geq h_{d-1}$.

Then there is a triangulated $(d-1)$-ball with h-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.

2.5 Shellable Ball Construction

In this section we strengthen the results of Theorems 2.15 and 2.18 by showing that in the situations of these theorems there is actually a *shellable* ball with the desired h-vector. The proof is independent of the proofs of these two previous theorems, though the overall setup is similar and we use the same notation for the h_i, g_i, and J. In particular, once again define g_i to be the vector from the first condition of Theorem 2.15 or Theorem 2.18, depending on the parity of d. When $d-1$ is even we have

$$(1, g_1, g_2, \ldots, g_{(d-3)/2}, g_{(d-1)/2}) :=$$
$$(1, h_1 - 1, h_2 - h_1, \ldots, h_{(d-3)/2} - h_{(d-5)/2}, \max\{h_{(d-1)/2} - h_{(d-3)/2}, 0\}).$$

Define J to be the compressed order ideal of monomials that has degree sequence $(1, g_1, g_2, \ldots, g_{(d-1)/2}, 0, 0, \ldots, 0)$ and let $J_{\leq j} = \cup_{i=0}^{j} J_i$. We once again use the Billea and Lee construction to build the ball $B(J)$ and find the desired ball as a subcomplex of the boundary of $B(J)$. To show that the desired ball is shellable we work with the ball directly (rather than its complement) and use a more involved inductive argument.

Theorem 2.19. *Let $(1, h_1, h_2, \ldots, h_{d-1}, 0)$ satisfy the conditions of Theorem 2.15 or Theorem 2.18. Then there is a shellable $(d-1)$-ball with h-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.*

Proof. We describe below the case where $d-1$ is even; the other case is handled similarly.

Following the Billera and Lee construction, build the d-ball $B(J)$ that has h-vector $(1, g_1, g_2, \ldots, g_{(d-1)/2}, 0, 0, \ldots, 0)$. We show that in the boundary sphere $\partial B(J)$ there is a shellable ball with the desired h-vector.

We split the proof into two cases depending on the sign of $h_{(d-1)/2} - h_{(d-3)/2}$. In both cases, we start by inductively building a shellable ball $\mathcal{C}(J)$ in $\partial B(J)$ that obtains the first half of the desired h-vector. We then add to this shelling order an additional collection of facets of $\partial B(J)$ to obtain the complete h-vector.

The Case $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$.

We begin with the case $h_{(d-1)/2} - h_{(d-3)/2} \geq 0$. Our first goal is to construct a shellable $(d-1)$-ball $\mathcal{C}(J)$ in $\partial B(J)$ with

$$h(\mathcal{C}(J)) = (1, h_1, h_2, \ldots, h_{(d-1)/2}, 0, 0, \ldots, 0).$$

Fix any rev-lex initial segment I of the monomials of J. Note that I is a compressed order ideal. Let $(c_0 = 1, c_1, \ldots, c_{(d-1)/2})$ be the degree sequence of I. The d-ball $B(I)$ has h-vector $(1, c_1, \ldots, c_{(d-1)/2}, 0, 0, \cdots, 0)$. Using induction on the number of monomials in I we construct a shellable $(d-1)$-ball $\mathcal{C}(I)$ in $\partial B(I)$ with h-vector

$$h(\mathcal{C}(I)) = (1, 1 + c_1, 1 + c_1 + c_2, \cdots, \sum_{i=0}^{(d-1)/2} c_i).$$

The case $I = J$ constructs the desired ball $\mathcal{C}(J)$.

The Facets of $\mathcal{C}(I)$.

We now describe the facets of $\mathcal{C}(I)$. Pick any monomial $m \in I$. We can write $\alpha^{-1}(m)$ as

$$\alpha^{-1}(m) = \cup_{j=1}^{(d+1)/2} \{i_j, i_{j+1}\}.$$

For $k \in \{1, 2, \ldots, (d+1)/2\}$, the facets of $\mathcal{C}(I)$ in $\alpha^{-1}(m)$ are the sets in $\partial B(I) \cap \alpha^{-1}(m)$ of the form

$$\left(\cup_{1 \leq j \leq (d+1)/2, j \neq k} \{i_j, i_{j+1}\}\right) \cup \{i_{k+1}\}. \tag{2.6}$$

In other words, starting with $\alpha^{-1}(m)$, if we remove the first element of any pair of vertices and the resulting face is in $\partial B(I)$, then the face is a facet of $\mathcal{C}(I)$.

Note that for any face F described in (2.6) there are exactly two monomials m, m' such that $\alpha^{-1}(m)$ and $\alpha^{-1}(m')$ contain F. Let $m <_{rl} m'$. The face F is contained in $\partial B(I)$, and hence in $\mathcal{C}(I)$, for exactly those ideals I that contain m but not m'.

Next we describe the restriction face for each facet of $\mathcal{C}(I)$. We can write any facet F of $\mathcal{C}(I)$ as $F = \cup_{j=1}^{d}\{p_j\}$ where $0 < p_1 < p_2 < \cdots < p_d$. Let $L(F)$ be the left endset of F, $L(F) := \{k : p_k = k\}$. Note that $L(F)$ may be empty. The restriction face $r(F)$ in our shelling order will be $\{p_k : k = |L(F)| + 2n, n \in \mathbb{N}\}$. Equivalently, starting after the left endset, every second vertex of F is in $r(F)$.

The Shelling Order for $\mathcal{C}(I)$.

Finally we describe the order for the shelling of $\mathcal{C}(I)$. To do this we write each facet F of $\mathcal{C}(I)$ as a disjoint union

$$F = L(F) \cup \left(\cup_{i=1}^{z_F} K_i(F)\right).$$

Each set $K_i(F)$ is a non-empty contiguous set of vertices $\{a, a+1, \ldots, b-1, b\}$ such that $a-1$ and $b+1$ are not in F. The $K_i(F)$ are chosen so that for $i < j$ any element of $K_i(F)$ is less than any element of $K_j(F)$.

Let F and G be distinct facets of $\mathcal{C}(I)$.

If $|L(F)| > |L(G)|$ we place F before G in our shelling order. This is equivalent to ordering the facets by the sizes of their restriction faces in increasing order.

If $|L(F)| = |L(G)|$, let a and b be the smallest elements of $K_1(F)$ and $K_1(G)$, respectively. If $a < b$ we place F before G in the shelling.

If $a = b$, our ordering depends on the parity of $|K_1(F)|$ and $|K_1(G)|$. When $|K_1(F)|$ is odd and $|K_1(G)|$ is even we place F before G in the shelling.

If both $|K_1(F)|$ and $|K_1(G)|$ are odd and $|K_1(F)| < |K_1(G)|$, place F before G in the shelling. If $|K_1(F)| = |K_1(G)|$ is odd, let e be the largest vertex that is in either F or G but not both. Place the facet that contains e earlier in the shelling.

If both $|K_1(F)|$ and $|K_1(G)|$ are even and $|K_1(F)| < |K_1(G)|$, place G before F in the shelling. If $|K_1(F)| = |K_1(G)|$ is even, then repeat the above steps on $K_2(F)$ and $K_2(G)$ rather than the K_1's. Continuing in this manner, since F and G are distinct there is some i such that $K_i(F) \neq K_i(G)$, which results in a way to order F and G.

Proof that we have a Shelling Order of $\mathcal{C}(I)$.

Now we inductively show that the above facets, restriction faces, and ordering give a shelling order for $\mathcal{C}(I)$. For the base case $I = \{1\}$ we have $B(I) = \alpha^{-1}(1) = \{i\}_{i=1}^{d+1}$. The above described ordering gives a shelling $\{F_i\}_{i=1}^{(d+1)/2}$ where

$$F_i = \bigcup_{1 \leq j \leq d+1, j \neq d-2i+2} \{j\}.$$

It is straightforward to check that this is a shelling order with restriction faces $r(F_i)$ as defined above and the desired h-vector.

We now consider the step where we add a new monomial to our ideal I and a corresponding new facet to our d-ball. Let I be the previous ideal and let $I' = I \cup \{M\}$ be our new ideal. So $\alpha^{-1}(M)$ is the new facet we add to get the ball $B(I')$. Let $s(M)$ be the degree of M. Note that $s(M)$ is the number of shifted pairs of $\alpha^{-1}(M)$. Hence we write $\alpha^{-1}(M)$ as a disjoint union

$$\alpha^{-1}(M) = \left(\cup_{i=1}^{d+1-2\cdot s(M)}\{i\}\right) \cup \left(\cup_{j=1}^{s(M)}\{i_j, i_j+1\}\right) \tag{2.7}$$

where $i_{j+1} > i_j + 1$ for every j.

We now determine the differences between the sets of facets of $\mathcal{C}(I)$ and $\mathcal{C}(I')$. Any facet of $\mathcal{C}(I)$ is also in $\mathcal{C}(I')$ unless the addition of $\alpha^{-1}(M)$ to $B(I)$ causes the facet to no longer be on the boundary of $B(I')$. Hence the facets in $\mathcal{C}(I)$ but not $\mathcal{C}(I')$ are

$$\left(\cup_{i=1}^{d+1-2\cdot s(M)}\{i\}\right) \cup \left(\cup_{1\leq j\leq s(M), j\neq k}\{i_j, i_j+1\}\right) \cup \{i_k\} \tag{2.8}$$

for $k = 1, 2, \ldots, s(M)$. The facets in $\mathcal{C}(I')$ but not $\mathcal{C}(I)$ are just the facets of $\mathcal{C}(I')$ in $\alpha^{-1}(M)$. In particular, the facets with left endset of size $d+1-2\cdot s(M)$ that are in $\mathcal{C}(I')$ but not $\mathcal{C}(I)$ are

$$\left(\cup_{i=1}^{d+1-2\cdot s(M)}\{i\}\right) \cup \left(\cup_{1\leq j\leq s(M), j\neq k}\{i_j, i_j+1\}\right) \cup \{i_k+1\} \tag{2.9}$$

where $k = 1, 2, \ldots, s(M)$.

We show that the facets F of $\mathcal{C}(I')$ ordered as described above are a shelling order of $\mathcal{C}(I')$ with the claimed restriction faces.

No facets F with $|L(F)| > d+1-2\cdot s(M)$ are added or removed by moving from $\mathcal{C}(I)$ to $C(I')$. Therefore, the first part of the shelling order of $\mathcal{C}(I')$ is identical to that of $\mathcal{C}(I)$. We only need to check that the restriction faces are as claimed for the facets of $\mathcal{C}(I')$ with left endset of size less than or equal to $d+1-2\cdot s(M)$. We do this in four claims depending on the size of $L(F)$ and if $F \in \mathcal{C}(I)$.

Claim 1. *Let F be a facet in $\mathcal{C}(I')\backslash\mathcal{C}(I)$ with $|L(F)| = d+1-2\cdot s(M)$. Then $r(F)$ is the unique minimal new face of F in our ordering of the facets of $\mathcal{C}(I')$.*

Proof of Claim 1. To show that $r(F)$ is the unique minimal new face we show that it is a new face and that for each vertex $v \in r(F)$ the face $F - \{v\}$ is in some previous facet of our shelling order.

Comparing (2.8) and (2.9), the restriction face $r(F)$ is equal to $r(G)$ for some $G \in \mathcal{C}(I)\backslash\mathcal{C}(I')$. Therefore, $r(F)$ cannot be contained in some face $H \in \mathcal{C}(I')$ with $|L(H)| > d+1-2\cdot s(M)$. For $H \in \mathcal{C}(I')$ with $|L(H)| = d+1-2\cdot s(M)$, one can check exhaustively that if H contains $r(F)$ then H is after F in our ordering. This exhaustive check uses the fact that exactly one of the monomials n such that $\alpha^{-1}(n)$ contains H must be rev-lex less than M. This completes the proof that $r(F)$ is a new face.

From the inductive assumption we know that $\mathcal{C}(I)$ is shellable with the above described shelling order and restriction faces. For each vertex $v \in r(F)$, the face $F - \{v\}$ is either in one of the facets of $\mathcal{C}(I')\backslash\mathcal{C}(I)$ that is before F in the shelling order or is equal to $G - \{w\}$ where G is one of the facets in $\mathcal{C}(I)\backslash\mathcal{C}(I')$ and $w \in r(G)$. These two cases are distinguished by whether the vertex v comes after (or in) the one contiguous set in F of odd length or before the odd contiguous set.

In the former case we immediately have that $F - \{v\}$ is in some previous facet of our shelling order. In the latter case we know that there is some facet G' in $\mathcal{C}(I)$ before G in the shelling order such that $G - \{w\} \subseteq G'$. It is straightforward to check that the structure of the facets of $\mathcal{C}(I)\backslash\mathcal{C}(I')$ forces $G' \in \mathcal{C}(I')$. The fact that all of the facets of $\mathcal{C}(I')\backslash\mathcal{C}(I)$ come later in our ordering than the facets of $\mathcal{C}(I)\backslash\mathcal{C}(I')$ shows that $F - \{v\}$ is in some previous facet of our shelling order, finishing Claim 1. $\square$

Claim 2. *Let F be a facet in $\mathcal{C}(I')\cap\mathcal{C}(I)$ with $|L(F)| = d+1-2\cdot s(M)$. Then $r(F)$ is the unique minimal new face of F in our ordering of the facets of $\mathcal{C}(I')$.*

Proof of Claim 2. By the inductive hypothesis we know that $r(F)$ is a new face in the shelling order on $\mathcal{C}(I)$. We need to check $r(F)$ is not contained in a facet of $\mathcal{C}(I')\backslash\mathcal{C}(I)$ that appears before F in our ordering on $\mathcal{C}(I')$. Once again, this is done by exhausting the possible cases, keeping in mind that all of the facets of $\mathcal{C}(I')$ are on the boundary of $B(I')$.

To complete the proof of Claim 2 we must show that for each vertex $v \in r(F)$ the face $F - \{v\}$ is in some facet of $\mathcal{C}(I')$ that is before F in our ordering. Let $G' := F - \{v\}$. Choose the first facet G in the shelling order on $\mathcal{C}(I)$ such that $G' \subseteq G$. By the inductive hypothesis we know that such a G exists and that G is before F in the shellling order of $\mathcal{C}(I)$. If $G \in \mathcal{C}(I')$ we are done, so assume $G \in \mathcal{C}(I)\backslash\mathcal{C}(I')$. Write G in the form (2.8) above.

If G' is obtained from G by removing $\{i_k\}$, then any vertex w that we add to G' would either be in $L(G' \cup \{w\})$ or an even numbered vertex after the first gap in $G' \cup \{w\}$. Hence $w \notin r(G' \cup \{w\})$. Therefore, G' cannot be equal to $F - \{v\}$, and this case does not occur.

If G' is obtained from G by removing $\{i_j + 1\}$ for $j < k$ or $\{i_j\}$ for $j > k$, then G' does not contain $r(G)$. Therefore G' is contained in some facet of $\mathcal{C}(I)$ before G in our ordering, contradicting our minimality assumption on G.

If G' is obtained from G by removing $\{i_j + 1\}$ for $j > k$, then for every vertex w we can show that $G' \cup \{w\}$ is not equal to F for (at least) one of the following reasons.

1. $G' \cup \{w\}$ is not of the form (2.6).
2. $G' \cup \{w\}$ is contained in $\alpha^{-1}(n)$ and $\alpha^{-1}(n')$ for two distinct monomials n, n' both rev-lex less than or equal to M, and hence is not in $\partial B(I')$.

If G' is obtained from G by removing $\{i_j\}$ for $j < k$, then for every vertex w either

1. $G' \cup \{w\}$ is not equal to F for one of the two reasons above,
2. $G' \cup \{w\}$ is not equal to F because $G' \cup \{w\}$ is in $\mathcal{C}(I') \backslash \mathcal{C}(I)$, or
3. there is a facet $H \in \mathcal{C}(I') \backslash \mathcal{C}(I)$ such that $G' \subseteq H$ and H is before $G' \cup \{w\}$ in our ordering.

This completes Claim 2. $\square$

Now we consider the latter part of our shelling consisting of faces F such that $|L(F)| < d + 1 - 2 \cdot s(M)$. For every face F in $\mathcal{C}(I) \backslash \mathcal{C}(I')$, $|L(F)| = d + 1 - 2 \cdot s(M)$. Thus no faces are removed in the later part of the shelling. For each even k such that $0 \leq k < d + 1 - 2 \cdot s(M)$

there is one face F_k in $\mathcal{C}(I')\backslash\mathcal{C}(I)$ with $|L(F_k)| = k$. Using the notation of (2.7) for $\alpha^{-1}(M)$ this new face is given by

$$F_k = \left(\cup_{i=1}^{d+1-2\cdot s(M)}\{i\}\right)\backslash\{k+1\} \cup \left(\cup_{j=1}^{s(M)}\{i_j, i_j+1\}\right).$$

Claim 3. *Let F_k be a facet in $\mathcal{C}(I')\backslash\mathcal{C}(I)$ with $|L(F_k)| < d+1-2\cdot s(M)$. Then $r(F_k)$ is the unique minimal new face of F_k in our ordering of the facets of $\mathcal{C}(I')$.*

Proof of Claim 3. Note that

$$r(F_k) = \left(\cup_{i=k/2}^{d/2-s(M)-3/2}\{2i+3\}\right) \cup \left(\cup_{j=1}^{s(M)}\{i_j\}\right).$$

Fix k and let G be a facet of $\mathcal{C}(I')$ such that $r(F_k) \subseteq G$. If $|L(G)| > k$, then because I' is a compressed order ideal G would already be in two facets of $B(I')$ and hence would not be in $\partial B(I')$. If $|L(G)| = k$, then by checking all possible choices for G we see that G must be after F_k in our ordering. If $|L(G)| < k$, then by the definition of our ordering G is after F_k. Therefore, $r(F_k)$ is a new face.

For $k/2 \le i \le d/2 - s(M) - 3/2$, the face $F_k - \{2i+3\}$ is in F_{2i+2}, which is before F_k in our ordering of the facets of $\mathcal{C}(I')$. Any face of F_k that does not contain $\cup_{j=1}^{s(M)}\{i_j\}$ is contained in one of the facets of $\mathcal{C}(I')\backslash\mathcal{C}(I)$ with $|L(F_k)| = d+1-2\cdot s(M)$. Therefore $r(F_k)$ is the unique minimal new face of F_k, as desired. □

Claim 4. *Let F be a facet in $\mathcal{C}(I')\cap\mathcal{C}(I)$ with $|L(F)| < d+1-2\cdot s(M)$. Then $r(F)$ is the unique minimal new face of F in our ordering of the facets of $\mathcal{C}(I')$.*

Proof of Claim 4. The fact that $r(F)$ is a new face is proved as in Claim 2.

Now let $v \in r(F)$ and consider the face $F' := F - \{v\}$. We need to show that $F' \subseteq G$ for some $G \in \mathcal{C}(I')$ such that G is before F in our ordering. By the inductive hypothesis we know that there is an $H \in \mathcal{C}(I)$ such that $F' \subset H$ and H is before F in our ordering. The result is trivial if $H \in \mathcal{C}(I')$, so assume $H \in \mathcal{C}(I)\backslash\mathcal{C}(I')$.

If $H\backslash F'$ were an even number in $L(H)$, then $|L(F)|$ would be odd, contradicting the form of the elements of $\mathcal{C}(I)$. If $H\backslash F' = \{k+1\}$ is an odd number in $L(H)$, then since $v \in r(F)$ and $F \in \mathcal{C}(I')$ we must have $v = d+2-2\cdot s(M)$. Hence the facet $F_k \in \mathcal{C}(I')\backslash\mathcal{C}(I)$ contains F'. Also, F_k comes before F in our ordering on $\mathcal{C}(I')$ because of the parity of each facet's leftmost contiguous set (that is not the left endset).

When $H\backslash F'$ is not in $L(H)$ there exists a face $H' \in \mathcal{C}(I')\backslash\mathcal{C}(I)$ with $|L(H')| = d+1-2\cdot s(M)$ such that $F' \subset H'$. The first contiguous set where F and H' disagree starts at the same vertex but has even length for F and odd length for H'. Therefore H' is before F in the ordering of $\mathcal{C}(I')$, completing Claim 4. $\square$

We have now shown that $C(I')$ is shellable with the claimed order and restriction faces. Using this shelling, it is straightforward to check inductively that $C(I')$ has the desired h-vector.

Construction of the Second Half of the Shelling Order.

Next we add additional facets to our shelling of $\mathcal{C}(J)$ to complete the second half of the desired h-vector.

Given a facet $\alpha^{-1}(m)$ of $B(J)$, the codimension-one faces obtained by removing $\{k\}$ where k is even and $1 < k \leq d+1-2\cdot s(m)$ are facets of $\partial B(J)$. These are the facets we add during the second half of the shelling. For a fixed k, call the collection of all such facets A'_k. For each facet $F \in A'_k$ used in the second half of our shelling

$$r(F) = \left(\cup_{i=1}^{k-1}\{i\}\right) \cup \left(\cup_{i=k/2}^{(d-1)/2-s(m)}\{2i+1\}\right) \cup \left(\cup_{j=1}^{s(m)}\{i_j\}\right).$$

For any fixed facet $\alpha^{-1}(M)$ of $B(J)$ we do not necessarily add every face of the form $A'_k \cap \alpha^{-1}(M) = \alpha^{-1}(M)\backslash\{k\}$ to our shelling. However, if we choose not to add some face $A'_k \cap \alpha^{-1}(M)$ to our shelling, then any other face $A'_l \cap \alpha^{-1}(M)$ with $l > k$ also cannot be added to the shelling. Also, for $M' \in J$ rev-lex before M we do not allow the addition of any face $A'_l \cap \alpha^{-1}(M')$ such that $l \geq k$.

For each facet F we are adding to the second half of our shelling there is a unique monomial $m \in J$ such that $F \in \alpha^{-1}(m)$. We order our facets by the rev-lex order on the corresponding monomials in J with the facets corresponding to the rev-lex larger monomials coming earlier in our ordering. We order facets that have the same corresponding monomials by increasing size of their restriction faces. Equivalently, for $k > l$ the element of A'_l in $\alpha^{-1}(m)$ comes before the element of A'_k. We show that under these conditions we get a shelling order than extends the shelling of $\mathcal{C}(J)$.

Proof that the Extended Ordering is a Shelling Order.

Let $F \subset \alpha^{-1}(M)$ be a face added in the second half of the shelling. We first show that $r(F)$ is a new face.

The only faces of the form (2.6) that can contain $r(F)$ are contained in $\alpha^{-1}(n)$ and $\alpha^{-1}(n')$ for distinct monomials n, n' both rev-lex before M and hence are not in $\partial B(I)$. Therefore $r(F)$ is not contained in any facet from the first half of the shelling. Also note that the only facets of $B(J)$ that contain $r(F)$ are equal to $\alpha^{-1}(m')$ for a monomial m' that is rev-lex less than or equal to M. The facets of the form $\alpha^{-1}(M) \cap A'_k$ with a smaller restriction face than that of F do not contain $r(F)$. Therefore, no facet in the second half of the shelling but before F in our ordering contains $r(F)$. Thus $r(F)$ is a new face.

Now let $v \in r(F)$. We show that $F' := F - \{v\}$ is contained in some facet before F in our shelling.

If $v < k$ and v is even, then the face $F' \cup \{k\}$ is a codimension-one face of $\alpha^{-1}(M)$ and is in our shelling by the conditions on restriction face size. This face gives the desired codimension-one intersection.

If $v < k$ and v is odd we have two cases.

1. The facet $F' \cup \{k\}$ is a facet in our shelling, proving the desired result.
2. If $F' \cup \{k\}$ is not in $\mathcal{C}(J)$, then $F' \cup \{k\} \cup \{w_1\}$ is a facet of $B(J)$ where w_1 is the smallest number that is greater than $d+1-2 \cdot s(M)$ and not in F. Therefore the face $F' \cup \{w_1\}$ is in $B(J)$. If $F' \cup \{w_1\}$ is in $\mathcal{C}(J)$ then we are done. If not we let w_2 be the smallest number greater than w_1 that is not in F. We must have $F' \cup \{w_1\} \cup \{w_2\}$ in $B(J)$. If $F' \cup \{w_2\}$ is in $\mathcal{C}(J)$ we are done. If not we repeat the above process, replacing w_1 with a new w_3 and so on, until we reach a facet that is in $\mathcal{C}(J)$ and contains F'.

If $v \in r(F)$ and $v > k$ we have two cases.

1. The facet $F' \cup \{k\}$ is a facet in our shelling, and we are done with this step.
2. If $F' \cup \{k\}$ is not in $\mathcal{C}(J)$, then the face $F' \cup \{k\} \cup \{w\}$ must be in $B(J)$ where w is the smallest number that is greater than v and is not in F. The facet $F' \cup \{k\} \cup \{w\}$ is after $\alpha^{-1}(M)$ in the shelling order on $B(J)$ induced by the rev-lex ordering on J. Hence, by our conditions on which facets must be added to the second half of our shelling, $F' \cup \{w\}$ must be a facet in the second half of the shelling and before F in our ordering, completing the desired claim in this case.

Proof that All of the Desired h-vectors are Obtained.

Finally we show that we get all of the claimed h-vectors using this construction. Let A'_k be as above. For each face $F \in A'_k$ we know $F \cup \{k\}$ is an element of $B(J)$ and corresponds to a monomial in J. Order the elements of A'_k by the rev-lex order on these corresponding monomials with the rev-lex largest monomials first. Let A_k be the first $h_{(d+k-1)/2}$ elements of A'_k using this ordering. Let M_k be the rev-lex smallest monomial such that $\alpha^{-1}(M_k)$ contains an element of A_k.

For every even $k \geq 2$ we must show that $M_{k+2} \geq_{rl} M_k$. Note that for any monomial $m \in J$, the face $\alpha^{-1}(m)$ contains an element of A'_k if and only if $m \in J_{\leq (d-k+1)/2}$. Therefore, if J does not contain any monomials of degree $(d-k+1)/2$ that are rev-lex larger than M_{k+2}, then since $h_{(d+k-1)/2} \geq h_{(d+(k+2)-1)/2}$ we know that $M_k \leq_{rl} M_{k+2}$.

If J contains a monomial of degree $(d-k+1)/2$ that is rev-lex larger than M_{k+2}, then J must contain all of the monomials of degree less than or equal to $(d-k+1)/2$ that are rev-lex less than M_{k+2}. Since each difference $h_{(d-l+1)/2} - h_{(d+l-1)/2}$ counts how many elements of each A'_l are not selected for A_l, it is sufficient to show that there are at least $h_{(d-k+1)/2} - h_{(d+k-1)/2}$ monomials in $J_{\leq (d-k+1)/2}$ that are rev-lex less than M_{k+2}.

Since the differences $h_{(d-l+1)/2} - h_{(d+l-1)/2}$ form an M-vector we know that

$$\left(h_{(d-(k+2)+1)/2} - h_{(d+(k+2)-1)/2}\right)^{<(d-(k+2)+1)/2>} \geq h_{(d-k+1)/2} - h_{(d+k-1)/2}.$$

From the definition of M_{k+2} the number of monomials in $J_{\leq (d-(k+2)+1)/2}$ that are rev-lex less than M_{k+2} is the number of elements of $A'_{k+2} \backslash A_{k+2}$, namely $h_{(d-(k+2)+1)/2} - h_{(d+(k+2)-1)/2}$. Therefore, the desired result follows from showing that the $((d-(k+2)+1)/2)$th pseudopower of the number of monomials in $J_{\leq (d-(k+2)+1)/2}$ that are rev-lex less than M_{k+2} is equal to the number of monomials in $J_{\leq (d-k+1)/2}$ that are rev-lex less than M_{k+2}. This is a consequence of Claim 5.

Claim 5. *Fix $q > 0$ and let M be a monomial of degree less than or equal to $q-1$. Let I be the compressed order ideal of all monomials of degree less than or equal to q that are rev-lex less than M. Let $(1, n_1, n_2, \dots, n_q)$ be the degree sequence of I. Then*

$$\left(\sum_{i=0}^{q-1} n_i\right)^{<q-1>} = \sum_{i=0}^{q} n_i.$$

Proof of Claim 5. We construct the compressed order ideal I' with degree sequence $(1, 1+n_1, 1+n_1+n_2, \dots, \sum_{i=0}^{q} n_i)$ and show that no additional degree q monomials can be added to this order ideal, which proves the claim.

Let $m = \prod_{j=1}^{a} X_{i_j}$ be a monomial in I, where a is the degree of m. Then for $1 \leq l \leq q-a$ let $m_l = X_1^l \prod_{j=1}^{a} X_{i_j+1}$. Define I' to be the union of all of the m_l for $m \in I$. Then I' has the desired degree sequence, and it is straightforward to check that I' is a compressed order ideal.

Let A be the degree of M. Since I contains all monomials of degree less than or equal to q that are rev-lex less than M, M_{q-A} is the rev-lex smallest monomial of degree q that is not in I'. Since M has degree less than q, the monomial M_{q-A} contains a non-zero power of X_1. If we could add M_{q-A} to I' and still have an order ideal, then M_{q-A}/X_1 would be in I', which would imply that $M_{q-A} \in I'$. Therefore M_{q-A} cannot be added to I', proving the claim. □

The Case $h_{(d-1)/2} - h_{(d-3)/2} < 0$.

We now consider the case where $h_{(d-1)/2} - h_{(d-3)/2} < 0$. In this case $g_{(d-1)/2} = 0$, hence

$$h(\partial B(I)) = (1, h_1, h_2, \dots, h_{(d-5)/2}, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-3)/2}, h_{(d-5)/2}, \dots, h_2, h_1, 1).$$

Let J be the compressed order ideal with degree sequence $(1, g_1, g_2, \ldots, g_{(d-3)/2})$. Using the same argument as for the previous case we build a shellable $(d-1)$-ball $\mathcal{C}(J)$ with h-vector $(1, h_1, h_2, \ldots, h_{(d-3)/2}, 0, \ldots, 0)$.

To obtain a ball with the desired h-vector we make a slight alteration to the second half of the shelling described in the previous case. Note that for each monomial $m \in J$ the face $\alpha^{-1}(m)\backslash\{1\}$ is in $\partial B(J)$. Call the collection of all such facets A_1'. Order the elements of A_1' by the rev-lex order on the corresponding monomials of J with the facets corresponding to the rev-lex largest monomials first. Note that

$$|A_1'| = 1 + g_1 + g_2 + \cdots + g_{(d-3)/2} = h_{(d-3)/2} > h_{(d-1)/2}.$$

We therefore define A_1 to be the first $h_{(d-1)/2}$ elements of A_1'. For $k > 1$ define the A_k as in the previous case.

For any facet $F \in A_1$ and corresponding monomial $M \in J$ such that $F = \alpha^{-1}(M)\backslash\{1\}$ define

$$r(F) = \left(\cup_{i=1}^{(d-1)/2-s(M)}\{2i+1\}\right) \cup \left(\cup_{j=1}^{s(M)}\{i_j\}\right).$$

Then arguing as above, adding the faces in each of the A_i to the the ball $\mathcal{C}(J)$ results in a shellable ball with the desired h-vector. $\square$

2.6 Consequences of the Construction

As noted in the previous section, the conditions of Theorems 2.15 and 2.18 are not in general necessary for the existence of a ball with a given h-vector. However, in dimensions three and four it is straightforward to check that the conditions of Conjecture 2.4

imply the conditions of Theorem 2.18 and Theorem 2.15, respectively. Since we have already shown the necessity of the conditions of Conjecture 2.4 in dimensions three and four, these conditions give a complete characterization of the h-vectors of triangulated three- and four-balls. As mentioned previously, this result was first obtained by Lee and Schmidt in [17].

Starting in dimension five we know that the conditions of Conjecture 2.4 are no longer sufficient and we also know that the conditions of Theorems 2.15 and 2.18 are no longer necessary. In particular, our construction cannot create any five-balls with $h_2 < h_1$, even though many such balls exist. Given any five-ball we can attach to it a single five-simplex by gluing along a single codimension-one face of the boundary of each ball. This process adds one new vertex to the original ball. As described in Section 2.3.2, this increases the h_1-value of the original ball by one without changing any of the other entries of the h-vector. Repeating this process we can create many different balls with $h_2 < h_1$.

While there exist balls with $h_2 < h_1$ that do not arise from adding vertices to other balls as described above, we have so far been unable to find any five-ball whose h-vector cannot be realized by adding vertices to a ball constructed using Theorem 2.18. In fact, using the methods of Section 2.3 it can be shown that many of the 'small' h-vectors that cannot be obtained by adding vertices to balls constructed using Theorem 2.18 cannot be the h-vectors of five-balls. We therefore make the following conjecture.

Conjecture 2.20. *A vector* $\mathbf{h} = (1, h_1, h_2, h_3, h_4, h_5, 0)$ *is the h-vector of a five-ball if any only if there exists some integer* $m > 0$ *such that* $\mathbf{h} = (1, h_1 - m, h_2, h_3, h_4, h_5, 0)$ *satisfies the conditions of Theorem 2.18.*

If any two balls satisfying the condition in Conjecture 2.20 are joined along a single codimension-one face, the h-vector of the resulting ball still satisfies the conditions of the conjecture. However, it is not true that the conditions of Theorems 2.15 and 2.18 give all of the h-vectors of balls that cannot be split along some codimension-one face (i.e. all the balls Δ with $\beta_{n-d,n-d+1}(k(\Delta)) = 0$). As an example, combining the ball with h-vector (1,3,6,10,6,3,0) formed from the construction of Theorem 2.18 with the shellable ball with h-vector $(1, 2, 0, 0, 0, 0, 0)$ by gluing along two boundary faces gives a ball with h-vector (1,5,7,10,5,3,0) but no codimension-one face along which to split.

Beyond dimension five we know that Conjecture 2.4 does not give a description of the h-vectors of balls but we do not have any conjecture to replace it. Determining even just a conjectural description of the h-vectors of these higher dimensional balls remains an interesting open problem.

Chapter 3

f-vectors of Simplicial Posets Balls

3.1 Introduction

We now turn our attention to the case of simplicial posets and the question of characterizing the f-vectors of simplicial cell decompositions of balls. Many of the ideas from the case of simplicial complexes will be used again with appropriate modifications. However, in contrast to the case of triangulations, issues involving the parity of the number of facets appear when we deal with simplicial cell decompositions. New types of arguments are needed to deal with this question, both for proving necessary conditions and constructions.

Recall from Chapter 1 that a *simplicial poset* P is a finite poset containing a minimal element $\hat{0}$ such that for every $p \in P$ the closed interval $[\hat{0}, p]$ is a Boolean algebra. We denote by $\Gamma(P)$ the regular CW-complex such that P is the face poset of $\Gamma(P)$, and we identify the elements of P with the corresponding closed faces of $\Gamma(P)$.

We now formally develop the idea of the f- and h-vectors of a simplicial poset P, following the same structure as for simplicial complexes. The *ith face number* of $\Gamma(P)$, denoted $f_i(\Gamma(P))$, is the number of i-dimensional faces of $\Gamma(P)$. Equivalently, $f_i(\Gamma(P))$ is the number of elements $p \in P$ such that $[\hat{0}, p]$ is a Boolean algebra of rank $i + 1$. In particular, $f_{-1}(\Gamma(P)) = 1$, corresponding to the empty face in $\Gamma(P)$ or the element $\hat{0}$ in P. The *dimension* of $\Gamma(P)$ is the largest i such that $f_i(\Gamma(P))$ is non-zero. We define $f_i(P)$, the ith face number of the poset P, by $f_i(P) = f_i(\Gamma(P))$. If the poset P is clear from the context we often write f_i instead of $f_i(P)$ or $f_i(\Gamma(P))$.

Let $d-1$ be the dimension of P. We record all of the face numbers of P in a single vector $f(P) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ called the *f-vector* of P. Once again, it will often be easier to work with an equivalent encoding of the face numbers called the *h-vector*. The entries of the h-vector $(h_0, h_1, \ldots, h_d)$ are obtained from the face numbers by the relation

$$\sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}.$$

In the case where $\Gamma(P)$ is a simplicial complex, this definition of the h-vector of $\Gamma(P)$ agrees with the definition of the h-vector of a simplicial complex from the previous chapter. Just like for simplicial complexes, if P is a simplicial poset of dimension $d-1$ then $h_d = \sum_{i=0}^{d}(-1)^{d-i} f_{i-1} = (-1)^{d-1}\tilde{\chi}(\Gamma(P))$. In particular, if P is a $(d-1)$-ball, then $h_d = 0$. In many cases we will study the differences between consecutive entries of the h-vector. We therefore define $g_i := h_i - h_{i-1}$, with $g_0 := h_0 = 1$.

A significant area of study is characterizing the possible h-vectors of various types of simplicial posets. Complete characterizations are already known for Cohen-Macaulay posets (see Section 3.2.5) and spheres (Section 3.2.6). In this chapter we investigate the question of characterizing the h-vectors of posets P such that $\Gamma(P)$ is a ball.

We being our study in Section 3.2 by providing additional background about simplicial posets, including Stanley's extension of the idea of the face ring to this more general setting. In Sections 3.3, 3.4, and 3.5 we develop additional necessary conditions on the h-vectors of balls. Section 3.3 focuses on a version of the generalized Dehn-Sommerville for simplicial cell decompositions of manifolds. In analogy to the case of triangulations of balls, this allows us to transfer restrictions on the h-vectors of spheres to conditions on the h-vectors of balls, but these conditions are not sufficient to completely characterize the h-vectors of balls.

In Section 3.4 we show that when h_1 of the boundary sphere is zero there is a surjective map from the face ring of the ball modulo a linear system of parameters to the face ring of the boundary sphere modulo a related linear system of parameters. This result gives a series of new inequalities relating the entries of the h-vectors of the ball and the boundary sphere. We also present a recent generalization of this idea due to Murai that proves similar inequalities in the cases where some h_i of the boundary sphere is zero for $i > 1$.

Section 3.5 gives a number of conditions on the h-vector of a ball that force the sum of the entries of the h-vector to be even. All of these conditions require that some entry of h-vector of the ball is zero (besides h_d, which is always zero) and that some entry of the h-vector of the boundary sphere is also zero. The first two results in this section follow from counting arguments involving the incidences of facets and codimension-one faces of the ball. The other two conditions are derived by adding to our ball the cone over the boundary the ball, resulting in a sphere of the same dimension as the original ball. We then look at the restriction map from the face ring of this new sphere to the face ring our original ball and use this map to transfer known conditions on the sphere to conditions on the ball.

In Section 3.6 we use the idea of shelling orders of simplicial posets (see Section 3.2.2 and [5, Definition 4.1]) to construct simplicial cell decompositions of balls with specific h-vectors. We present two general constructions as well as a third result that yields additional h-vectors in dimension five. We conclude the chapter in Section 3.7 by combining all of these results to give a complete characterization of the h-vectors of simplicial cell decompositions of balls in all even dimesions as well as dimensions three and five.

3.2 Notation and Background

In this section we provide background on many of the ideas mentioned in the introduction, as well as some additional useful results of Masuda about simplicial poset spheres.

3.2.1 The Order Complex of a Poset

A *chain* in a poset P is a collection of elements $p_1, \ldots, p_n \in P$ such that $p_1 < p_2 < \cdots < p_n$. Let $\overline{P} = P - \{\hat{0}\}$. The *order complex* of $\overline{P}$, denoted $\Delta(\overline{P})$, is the simplicial complex whose vertices are the elements of $\overline{P}$ and whose faces are the sets of elements forming chains in $\overline{P}$. For a simplicial poset P, when we refer to the order complex of P we mean $\Delta(\overline{P})$.

For a simplicial poset P, the spaces $\Gamma(P)$ and $|\Delta(\overline{P})|$ are homeomorphic (in fact, $\Delta(\overline{P})$ is isomorphic to the barycentric subdivision of $\Gamma(P)$). Therefore, an alternative characterization of a simplicial cell decomposition of a manifold M is a simplicial poset P whose order complex has a geometric realization homeomorphic to M.

3.2.2 Shellings of $\Gamma(P)$

The *facets* of any CW-complex are the maximal faces with respect to inclusion. If P is a simplicial poset then the facets of $\Gamma(P)$ correspond to the maximal elements of P. A CW-complex is *pure* if all of its facets have the same dimension.

Consider the case where P is a simplicial poset and $\Gamma(P)$ is a pure complex of dimension $d-1$. A *shelling* of $\Gamma(P)$ is an ordering $F_1, F_2, \ldots, F_t$ of the (closed) facets of $\Gamma(P)$ such that for $j \geq 2$ the intersection $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is a non-empty union of (closed) *facets* of ∂F_j. This is equivalent to the definition of a shelling of a CW-complex used by Björner [5, Definition

4.1] specialized to the case of $\Gamma(P)$ for a simplicial poset P. Define the *restriction face* of F_k, denoted $\sigma(F_k)$, to be the set of vertices v of F_k such that the facet of ∂F_k not containing v is in $\cup_{i=1}^{k-1} F_i$. Then the entries of the h-vector of P are given by $h_j = |\{F_k : |\sigma(F_k)| = j\}|$. We also use $\sigma(F_k)$ to refer to the face of F_k containing exactly the vertices in the set $\sigma(F_k)$.

3.2.3 Cones of Posets

Given a simplicial poset P, the *cone* over P is the simplicial poset $P \times [1,2]$ where $[1,2]$ is the poset of two elements with $2 > 1$. More specifically, the elements of $P \times [1,2]$ are the ordered pairs (p,i) where $p \in P$ and $i \in \{1,2\}$ and the covering relations are:

- If p covers q in P, then (p,i) covers (q,i) in $P \times [1,2]$ for $i \in \{1,2\}$.
- For all $p \in P$, $(p,2)$ covers $(p,1)$ in $P \times [1,2]$.

Topologically, $\Gamma(P \times [1,2])$ is the cone over $\Gamma(P)$, with $(\hat{0},2)$ corresponding to the cone point. The dimension of $P \times [1,2]$ is one greater than that of P. A straightforward calculation shows that the h-vector of $P \times [1,2]$ is equal to that of P augmented by $h_{d+1} = 0$.

3.2.4 The Face Ring of a Simplicial Poset

We now describe Stanley's idea of the face ring of a simplicial poset [33]. Let k be an infinite field. Define $S := k[x_p : p \in \overline{P}]$ to be the polynomial ring over k with variables indexed by the elements of $\overline{P}$. We define a grading on S by letting the degree of x_p be one more than the dimension of the face in $\Gamma(P)$ corresponding to p. So $[\hat{0},p]$ is a Boolean algebra of rank equal to the degree of x_p.

For elements p and q in P the *meet* of p and q, denoted $p \wedge q$, is the largest element that is less than both p and q. In general, the meet of two elements of a poset need not be well-defined. However, when P is a simplicial poset, if p and q have a common upper bound r, then the interval $[\hat{0}, r]$ is a Boolean Algebra. In a Boolean Algebra all pairs of elements have meets, so $p \wedge q$ is well defined in $[\hat{0}, r]$ and hence also in P.

Define I_P to be the ideal of S generated by all elements of the form $x_p x_q - x_{p\wedge q} \sum_r x_r$ where $p, q \in P$ and the sum is over all minimal upper bounds r of p and q. In the case where p and q have no common upper bound in P, this reduces to the element $x_p x_q$. If the meet of p and q is $\hat{0}$ then we set $x_{p\wedge q} = 1$. Define the *face ring* of P to be $A_P := S/I_P$.

In the case where P is the face poset of a simplicial complex Δ, the face ring A_P is the same as the face ring of Δ defined in the previous chapter. To see this, note that when $P = P(\Delta)$, any two elements of P have either no common upper bound or a unique minimal common upper bound (known as the join and denoted $p \vee q$). Identifying the elements of P and Δ, the join of two elements of Δ is given by taking their union, if the union is an element of Δ. Using this one can show that for any face $F = \{v_1, \ldots, v_k\} \in \Delta$, the element $x_{v_1} \cdots x_{v_k} - x_F$ is in I_P, while if $\{v_1, \ldots, v_k\} \notin \Delta$ then $x_{v_1} \cdots x_{v_k} \in I_P$. Hence A_P is generated by the variables corresponding to vertices and all monomials corresponding to non-faces are part of the ideal by which we quotient to form the face ring. One can also check that I_P contains no additional relations on the variables corresponding to the vertices.

3.2.5 Cohen-Macaulay Simplicial Posets

A simplicial poset P is *Cohen-Macaulay* if its order complex $\Delta(\overline{P})$ is a Cohen-Macaulay simplicial complex (see Section 2.1.4 for the definition of a Cohen-Macaulay simplicial complex). Recall that by Theorem 2.1, all triangulations of balls and spheres are Cohen-Macaulay simplicial complexes. From Section 3.2.1 we know that if P is a simplicial cell decomposition of a ball or sphere then $\Delta(\overline{P})$ is a triangulation of a ball or sphere, respectively. Hence all simplicial poset balls and spheres are Cohen-Macaulay.

Stanley proved that a simplicial poset P is Cohen-Macaulay if and only if its face ring A_P is a Cohen-Macaulay ring [33, Corollary 3.7]. Using this result, Stanley [33, Theorem 3.10] showed that if Q is a Cohen-Macaulay simplicial poset, then $h_0(Q) = 1$ and $h_i(Q) \geq 0$ for $i \geq 1$ (he also proved that these are sufficient conditions to characterize the h-vectors of Cohen-Macaulay simplicial posets). Stanley's book [35] is a good reference for more information on Cohen-Macaulay rings and complexes.

Let $T = T_0 \oplus T_1 \oplus \cdots$ be a finitely generated graded algebra over the (infinite) field $k = T_0$. Recall from Section 2.1.4 that the *Hilbert function* of T is $F(T,i) := \dim_k T_i$ where $i \geq 0$. Let $d = \dim T$ (Krull dimension). Then a *linear system of parameters (l.s.o.p)* for T is a collection of elements $\theta_1, \ldots, \theta_d \in T_1$ such that T is finitely generated as a $k[\theta_1, \ldots, \theta_d]$-module. From [33, Theorem 3.10] we know that when P is a Cohen-Macaulay simplicial poset, $\dim A_P = \dim(\Gamma(P)) + 1$ and an l.s.o.p. for A_P exists. Further, when P is a Cohen-Macaulay simplicial poset and $\theta_1, \ldots, \theta_d$ is an l.s.o.p. for A_P we have $F(A_P/(\theta_1, \ldots, \theta_d), i) = h_i(P)$ [33, Theorem 3.8], the same result as for Cohen-Macaulay simplicial complexes.

3.2.6 Simplicial Poset Spheres

As mentioned in the introduction, the complete characterization of the possible h-vectors of simplicial cell decompositions of spheres is already known. Stanley conjectured the result and proved sufficiency in 1991 [33, Theorem 4.3, Remark 4] and necessity was shown by Masuda in 2005 [21, Corollary 1.2].

Theorem 3.1. *Let* $\mathbf{h} = (h_0, h_1, \ldots, h_d) \in \mathbb{N}^{d+1}$*. Then there exists a simplicial poset* P *with* $h(P) = \mathbf{h}$ *and* $\Gamma(P)$ *a sphere if and only if* $h_0 = 1$, $h_i = h_{d-i}$ *for* $0 \leq i \leq d$, *and either* $h_i > 0$ *for* $0 \leq i \leq d$ *or* $\sum_{i=0}^{d} h_i$ *is even.*

The equations $h_i = h_{d-i}$ are a generalized version of the Dehn-Sommerville relations for triangulated spheres. Because of this symmetry in the h-vector, whenever d is odd, $\sum_{i=0}^{d} h_i$ is always even and the final condition in Theorem 3.1 is automatically satisfied. In the case where d is even, the parity of $\sum_{i=0}^{d} h_i$ is equal to the parity of $h_{d/2}$. Note that in contrast to the simplicial complex case, there are no M-vector conditions on the h-vectors of simplicial poset spheres. In particular, the h-vector of a simplicial cell decomposition of a sphere can have zero entries.

In addition to this numerical result we often need some of the stronger statements that were used to prove the necessity of the claim. The following theorem is discussed on pages 343-344 of the original proof of necessity due to Masuda [21] and is Theorem 2 in the paper [23] published two years later by Miller and Reiner giving a simplified proof of Masuda's result.

Theorem 3.2. *Let* P *be a simplicial poset such that* $\Gamma(P)$ *is a* $(d-1)$*-sphere. If* $h_i(P) = 0$ *for some* i *strictly between zero and* d*, then for every subset* $V = \{v_1, \ldots, v_d\}$ *of the vertices of* $\Gamma(P)$*, the number of facets of* $\Gamma(P)$ *with vertex set* V *is even.*

Since $\sum_{i=0}^{d} h_i(P)$ is equal to the number of facets of $\Gamma(P)$, Theorem 3.1 follows from Theorem 3.2.

We also isolate a useful result embedded in the proof of Theorem 3.2. This proposition relates the parity of the number of facets on a vertex set to the product of the variables in the algebra A_P corresponding to those vertices.

Proposition 3.3. *Let P be a simplicial poset such that $\Gamma(P)$ is a $(d-1)$-sphere and let $V = \{v_1, \ldots, v_d\}$ be a subset of the vertices of $\Gamma(P)$. Let $\Theta = \theta_1, \ldots, \theta_d$ be an l.s.o.p. for A_P. If $x_{v_1} \cdots x_{v_d}$ is zero in A_P/Θ then there are an even number of facets of $\Gamma(P)$ with vertex set V.*

In the situation of this proposition, if $h_i(P) = 0$ for some i then $\dim_k(A_P/\Theta)_i = 0$. Therefore, for any set of vertices $V = \{v_1, \ldots, v_d\}$ of $\Gamma(P)$ we know that $x_{v_1} \cdots x_{v_i} = 0$ in A_P/Θ and hence $x_{v_1} \cdots x_{v_d} = 0$. In this way we see that Theorem 3.2 follows from Proposition 3.3.

3.3 The h-vector of the Boundary of a Simplicial Poset

The goal of this section is to relate the h-vector of a simplicial cell decomposition of a manifold with boundary to the h-vector of the boundary complex. In the case of balls, this will allow us to use Theorem 3.1 about the h-vectors of spheres to restrict the possible h-vectors of balls.

Our starting point is Theorem 2.3, which gives the desired relationship for the case of simplicial *complexes* whose geometric realizations are manifolds with boundary. This theorem was originally Theorem 2.1 in a paper by I.G. Macdonald [20]. The entire first section of Macdonald's paper is done in the generality of cell complexes and applies in our case. When Macdonald proves Theorem 2.1, the only property of simplicial complexes that he uses is the fact that for each simplex y in the complex the interval $[\hat{0}, y]$ in the face poset is a Boolean algebra. Since this fact is true for simplicial posets, Macdonald's result holds in this more general setting as well. Expressing his result in terms of h- and g-vectors we have the following theorem.

Theorem 3.4. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a manifold with boundary. Then*

$$h_{d-i}(P) - h_i(P) = \binom{d}{i}(-1)^{d-1-i}\tilde{\chi}(\Gamma(P)) - g_i(\partial\Gamma(P))$$

for all $0 \leq i \leq d$.

In the special case where $\Gamma(P)$ is a $(d-1)$-ball this reduces to the equation $h_i(P) - h_{d-i}(P) = g_i(\partial\Gamma(P))$. In particular, for $0 \leq j \leq d$ we have

$$h_j(\partial\Gamma(P)) = \sum_{i=0}^{j}(h_i(P) - h_{d-i}(P)). \tag{3.1}$$

Applying this result and the Dehn-Sommerville equations, in the case where d is odd we have

$$\sum_{i=0}^{d} h_i(P) \equiv h_{(d-1)/2}(\partial\Gamma(P)) \equiv \sum_{i=0}^{d-1} h_i(\partial\Gamma(P)) \mod 2. \tag{3.2}$$

Using this relationship we now give a first set of necessary conditions on the h-vectors of simplicial poset balls. As discussed in Section 3.2.5, any simplicial poset P such that $\Gamma(P)$ is ball is a Cohen-Macaulay poset. Combining Stanley's characterization of the h-vectors of Cohen-Macaulay posets with Theorem 3.1 and Equations (3.1) and (3.2) we have the following conditions.

Theorem 3.5. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball. Then $h_0(P) = 1$, $h_d(P) = 0$, $h_i(P) \geq 0$ for all i, and $\sum_{i=0}^{j}(h_i(P) - h_{d-i}(P)) \geq 0$ for $0 \leq j \leq \lfloor (d-1)/2 \rfloor$. Further, if d is odd and $\sum_{i=0}^{j}(h_i(P) - h_{d-i}(P)) = 0$ for some $0 \leq j \leq (d-1)/2$, then $\sum_{i=0}^{d} h_i(P)$ is even.*

Rephrasing this main ideas of this theorem, if P is a simplicial cell decomposition of a ball then

- $h_0(P) = 1$ and $h_d(P) = 0$,
- The entries of the h-vector of P are non-negative,
- The entries of the h-vector of the boundary sphere of $\Gamma(P)$ are non-negative, and
- If $\Gamma(P)$ has even dimension and one of the entries of the h-vector of the boundary sphere of $\Gamma(P)$ is zero, then the boundary sphere has an even number of facets.

3.4 Inequalities Relating a Ball and its Boundary

Consider a simplicial poset ball such that h_1 of the boundary sphere is zero. In the following we derive inequalities relating the h-vectors of the boundary sphere and the ball in this case. The idea of the argument follows that of a similar result by Stanley [34, Theorem 2.1] for the h-vectors of simplicial complexes.

One of the main tools in this proof is a useful characterization of linear systems of parameters for the ring A_P. Fix an ordering $\{v_1, \ldots, v_n\}$ of the vertices of $\Gamma(P)$. Let $\theta_1, \ldots, \theta_d$ be a collection of homogeneous degree-one elements of A_P. We can write each element of our collection as a linear combination of the x_{v_j}, $\theta_i = \sum_{j=1}^{n} \Theta_{i,j} x_{v_j}$. This gives a $d \times n$ matrix $\Theta_{i,j}$ whose rows correspond to the θ_i.

Let F be a facet of $\Gamma(P)$. Define Θ_F to be the $d \times d$ submatrix of $\Theta_{i,j}$ obtained by restricting to the columns corresponding to the vertices of F. Then we have the following characterization of the collections of degree one elements that are linear systems of parameters.

Lemma 3.6. *Let P be a simplicial poset and let $\theta_1, \ldots, \theta_d$ be a collection of homogeneous degree one elements of A_P. Then $\theta_1, \ldots, \theta_d$ is an l.s.o.p. for A_P if and only if* $\det(\Theta_F) \neq 0$ *for all facets F of* $\Gamma(P)$.

The only if part of the lemma was proved by Masuda [21, Lemma 3.1] and Miller and Reiner [23, p. 1051]. The if direction follows from Proposition 5 of Miller and Reiner's paper. In this proposition Miller and Reiner show that for an l.s.o.p. $\theta_1, \ldots, \theta_d$, the ring $A_P/(\theta_1, \ldots, \theta_d)$ is spanned k-linearly by the images of the x_G for all elements $G \in P$. The only property of the l.s.o.p. used in the proof is the non-zero determinant assumption in the above lemma.

Now let P be a $(d-1)$-ball. Then $\partial(\Gamma(P))$ is a $(d-2)$-sphere. If $h_1(\partial(\Gamma(P))) = 0$ we know that $\partial(\Gamma(P))$ has only $d-1$ vertices. Therefore every facet of $\partial(\Gamma(P))$ has the same vertex set. Let F be a facet of $\Gamma(P)$ such that a codimension-one face of F is in $\partial(\Gamma(P))$. Let v be the vertex of F that is not $\partial(\Gamma(P))$. Note that all of the vertices of $\Gamma(P)$ not in F are interior vertices.

Let $\theta_1, \ldots, \theta_d$ be an l.s.o.p. for A_P. By Lemma 3.6 we know that Θ_F has non-zero determinant. Thus the span of $\{\theta_1, \ldots, \theta_d\}$ contains some element $\theta' = x_v + \sum_{w \notin F} c_w x_w$ where the sum is over the vertices of $\Gamma(P)$ not in F and the c_w are constants in k. This allows us to choose a new l.s.o.p. $\theta'_1, \ldots, \theta'_d$ for A_P such that θ'_d is a linear combination of interior vertices of $\Gamma(P)$. By Lemma 3.6 we know that $\det(\Theta'_G) \neq 0$ for all facets G of $\Gamma(P)$.

Let Q be the face poset of $\partial\Gamma(P)$. Let $f : A_P \to A_Q$ be given by setting all variables corresponding to faces in $\Gamma(P) \backslash \partial\Gamma(P)$ equal to zero. Identify the l.s.o.p. $\theta'_1, \ldots, \theta'_d$ with its image under f in A_Q. Let H be any facet of $\partial\Gamma(P)$. The last row of Θ'_H is all zeros, so the

$(d-1)\times(d-1)$ minor given by the first $(d-1)$ rows must have a non-zero determinant. Again using Lemma 3.6 we see that $\theta'_1,\ldots,\theta'_{d-1}$ is an l.s.o.p. for $\partial\Gamma(P)$.

Therefore, f induces a degree preserving surjection

$$f : A_P/(\theta'_1,\ldots,\theta'_d) \to A_Q/(\theta'_1,\ldots,\theta'_{d-1}).$$

Hence $h_i(P) = F(A_P/(\theta'_1,\ldots,\theta'_d), i) \geq F(A_Q/(\theta'_1,\ldots,\theta'_{d-1}), i) = h_i(Q)$. Summarizing, we have proved the following theorem.

Theorem 3.7. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_1(\partial(\Gamma(P))) = 0$. Then $h_i(P) \geq h_i(\partial(\Gamma(P)))$ for all $i \geq 0$.*

Now consider the case of a ball P such that $h_n(\partial(\Gamma(P))) = 0$ for some $n > 0$. If we let $\{\theta_1,\ldots,\theta_d\}$ be a generically chosen set of linear forms, then Murai [26] noted that there is a surjection

$$g : A_P/(\theta_1,\ldots,\theta_{d-1},\theta_d^n) \to A_Q/(\theta_1,\ldots,\theta_{d-1}).$$

Using this surjection along with the fact

$$\dim_k(A_P/(\theta_1,\ldots,\theta_{d-1}))_l = h_0(P) + h_1(P) + \cdots + h_l(P)$$

Murai was able to prove the following generalization of the previous theorem.

Theorem 3.8. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_n(\partial(\Gamma(P))) = 0$. Then*

$$h_l(\partial P) \leq h_l(P) + h_{l-1}(P) + \cdots + h_{l-(n-1)}(P) \qquad \textit{for } l \geq n.$$

3.5 Parity Conditions on the Sum of the $h_i(P)$

When P is a simplicial poset ball of even dimension (so d is odd) by Theorem 3.5 we know that if any $h_k(\partial\Gamma(P))$ is zero then $\sum_{i=0}^{d} h_i(P)$ is even. For odd-dimensional balls the situation is not as simple. In this section we derive a series of different conditions under which the sum of the $h_i(P)$ must be even. All of these conditions involve some $h_k(\partial\Gamma(P))$ and some $h_j(P)$ being zero. In Section 3.6 we construct odd-dimensional balls where the h-vector of the boundary sphere has a zero entry and $\sum_{i=0}^{d} h_i(P)$ is odd.

The proofs in this section use Masuda's conditions describing when a set of d vertices of a sphere must be the set of vertices of an even number of facets. In Lemma 3.9 we apply Theorem 3.2 to the boundary sphere of the ball, while in Lemma 3.12 we apply Proposition 3.3 to the sphere formed by the union of the ball and the cone over the boundary of the ball

3.5.1 Conditions from Counting Arguments

Our first two examples of this new type of condition follow from counting arguments involving the faces of our complexes. The main idea in both proofs is the following connection between a zero in the h-vector of the boundary sphere and a parity condition on the incidences between facets and codimension-one faces.

Lemma 3.9. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_k(\partial\Gamma(P)) = 0$ for some k strictly between zero and $d-1$. Then every set of $d-1$ vertices of P is contained in an even number of facets (possibly zero).*

Proof. Let S be a set of $d-1$ vertices of $\Gamma(P)$. If S is not contained in any facet of $\Gamma(P)$ we are done. Otherwise, let F be a face of $\Gamma(P)$ with vertex set S. If F is an interior face of $\Gamma(P)$ then since $\Gamma(P)$ is a manifold there are exactly two facets of $\Gamma(P)$ that have F as a codimension-one face. If F is a boundary face of $\Gamma(P)$ then there is exactly one facet of $\Gamma(P)$ that has F as a codimension-one face. Further, since some $h_k(\partial\Gamma(P)) = 0$, by Theorem 3.2 the number of boundary faces of $\Gamma(P)$ with vertex set S is even. Since no single facet of $\Gamma(P)$ can have multiple faces with the same vertex set, the total number of facets of $\Gamma(P)$ that contain S is even. $\square$

Now consider the case where $\Gamma(P)$ is a ball, $h_1(P) = 0$, and $h_k(\partial\Gamma(P)) = 0$ for some k. By Lemma 3.9, any set of $d-1$ vertices of P is contained in an even number of facets. In terms of the face numbers, $h_1(P) = 0$ implies $f_0(P) = d$, meaning that all of the vertices of $\Gamma(P)$ are in every facet. Therefore, every set of $d-1$ vertices of P is in every facet and hence the total number of facets of $\Gamma(P)$ is even. Recalling that $\sum_{i=0}^{d} h_i(P)$ is the number of facets of $\Gamma(P)$ we have the following proposition.

Proposition 3.10. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_1(P) = 0$, and $h_k(\partial\Gamma(P)) = 0$ for some k strictly between zero and $d-1$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

We can extend the result of Proposition 3.10 to the case $h_2(P) = 0$ (instead of $h_1(P) = 0$) using a somewhat more involved argument based on the same ideas.

Proposition 3.11. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_2(P) = 0$, and $h_k(\partial\Gamma(P)) = 0$ for some k strictly between zero and $d-1$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

Proof. Pick a facet F_0 of $\Gamma(P)$ with vertex set $V = \{v_1, \ldots, v_d\}$. Let Δ_0 be the induced subcomplex on the vertex set V; Δ_0 consists of all of the faces of $\Gamma(P)$ whose vertices are contained in the set V. Note that Δ_0 contains at least $\binom{d}{2}$ edges.

Let F_1 be a facet of $\Gamma(P) - \Delta_0$ that intersects Δ_0 in a face of dimension $d-2$. Since $\Gamma(P)$ is a manifold, unless $\Gamma(P) = \Delta_0$ such a facet F_1 must exist. If $\Gamma(P) = \Delta_0$ then $\Gamma(P)$ has d vertices, so $h_1(P) = 0$ and we are in the case of Proposition 3.10. Otherwise, let w_1 be the vertex of F_1 not in V and let Δ_1 be the induced subcomplex of $\Gamma(P)$ on the vertex set $V \cup \{w_1\}$. There must be at least $d-1$ edges in $\Delta_1 - \Delta_0$ in order for the facet F_1 to exist.

We can continue to build our complex in this manner until we reach $\Delta_{h_1} = \Gamma(P)$. This results in a minimum of $\binom{d}{2} + h_1(P) \cdot (d-1)$ edges in our complex. However, since $h_2(P) = 0$ this is exactly the number of edges in $\Gamma(P)$. So we must have added the minimum number of edges at each step in our construction. In particular, for $1 \leq i \leq h_1$, all of the facets of Δ_i that contain w_i must have the same vertex set as F_i.

By Lemma 3.9 we know that every set of $d-1$ vertices of $\Gamma(P)$ is contained in an even number of facets. In particular, let S be a set of $d-1$ vertices of F_{h_1} that includes the vertex w_{h_1}. The facets that contain the vertices of S are exactly those facets whose vertex set equals the vertex set of F_{h_1}. Therefore, there must be an even number of facets on the vertex set of F_{h_1}.

Since we are only interested in the parity of the number of facets on each vertex set we can now ignore the contribution of the facets on the vertex set of F_{h_1} and repeat the above argument on the complex Δ_{h_1-1} and the facet F_{h_1-1}. Continuing in this manner we see that there are an even number of facets on all of the sets of d vertices of $\Gamma(P)$. Therefore $\Gamma(P)$ has an even number of facets, as desired. $\square$

3.5.2 The Cone Over the Boundary of $\Gamma(P)$

Let P be a simplicial poset such that $\Gamma(P)$ is a manifold with boundary and let Q be the face poset of $\partial(\Gamma(P))$. Define the cone over the boundary of $\Gamma(P)$ to be $SP := P \cup (Q \times [1,2])$ with each element $(q,1) \in (Q \times [1,2])$ identified with the element in P corresponding to q. The covering relations in SP are all of the covering relations in P along with all of the covering relations in $Q \times [1,2]$. In the case where $\Gamma(P)$ is a $(d-1)$-ball, $\Gamma(SP)$ is a $(d-1)$-sphere.

The face numbers of the new complex $\Gamma(SP)$ are given by

$$f_i(\Gamma(SP)) = f_i(\Gamma(P)) + f_{i-1}(\partial\Gamma(P))$$

for $-1 \le i \le d-1$, where $f_{-2}(\partial\Gamma(P))$ is interpreted as zero. Using this equality, a straightforward calculation shows that the elements of the h-vector of $\Gamma(SP)$ are given by

$$\begin{aligned} h_i(\Gamma(SP)) &= h_i(\Gamma(P)) + h_{i-1}(\partial\Gamma(P)) \\ &= h_i(\Gamma(P)) + \sum_{j=0}^{i-1}(h_j(\Gamma(P)) - h_{d-j}(\Gamma(P))), \end{aligned} \tag{3.3}$$

with the last equality by equation (3.1).

We now consider the relationship between the algebras A_P and A_{SP}. Let v be the cone point of SP; v is the vertex corresponding to $(\hat{0},2)$ in $Q \times [1,2]$. There is a natural surjective map $f : A_{SP} \to A_P$ given by setting all of the variables corresponding to faces containing v equal to zero. If $\Theta = \theta_1, \ldots, \theta_d$ is an l.s.o.p. for A_{SP}, then by Lemma 3.6 the image of Θ under f (which we also write as Θ) is an l.s.o.p. for A_P. Therefore, there is an induced map $f : A_{SP}/\Theta \to A_P/\Theta$ with kernel generated (modulo Θ) by monomials containing a variable corresponding to a face containing v. We use this map f to prove the following lemma.

Lemma 3.12. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball and $h_k(P) = 0$ for some k strictly between zero and d. Let $V = \{v_1, \ldots, v_{d-1}, v_d\}$ be a set of vertices of $\Gamma(P)$ such that v_d is an interior vertex. Then $\Gamma(P)$ has an even number of facets with vertex set V.*

Proof. Let Θ be an l.s.o.p. for A_P and let v be the cone point of SP as above. Let $m := x_{v_1} \cdots x_{v_k}$ be a monomial in $(A_{SP})_k$. Since the dimension of $(A_P/\Theta)_k$ is $h_k(P) = 0$ we know that m is in the kernel of the map $f : A_{SP}/\Theta \to A_P/\Theta$ defined above. Therefore, in A_{SP}/Θ we can write m as a linear combination of monomials each containing a variable corresponding to a face containing v. Since v_d is an interior vertex of $\Gamma(P)$, $x_{v_d} m$ is zero in A_{SP}/Θ. Thus by Proposition 3.3 we know that there must be an even number of facets of $\Gamma(SP)$ with vertex set V. Since the cone point v is not in V, the facets of $\Gamma(SP)$ with vertex set V are exactly the same as the facets of $\Gamma(P)$ with vertex set V, proving the desired result. □

We now apply Lemma 3.12 in two different cases, when $h_1(\partial\Gamma(P)) = 0$ and also when $h_1(\partial\Gamma(P)) = 1$ and $h_j(\partial\Gamma(P)) = 0$ for some $1 < j < d-1$.

3.5.3 The Case $h_1(\partial\Gamma(P)) = 0$

When $h_1(\partial\Gamma(P)) = 0$ and the h-vector of P has some zero entry (besides $h_d(P)$, which is always zero), we can use Lemma 3.12 to show that $\Gamma(P)$ has an even number of facets.

Proposition 3.13. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_1(\partial\Gamma(P)) = 0$, and $h_k(P) = 0$ for some $0 < k < d$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

Proof. Since $h_1(\partial\Gamma(P)) = 0$ we know that $\partial\Gamma(P)$ has only $d-1$ vertices. Therefore every facet of $\Gamma(P)$ contains an interior vertex. By Lemma 3.12 there are an even number of facets (possibly zero) on every set of d vertices of $\Gamma(P)$. Hence $\sum_{i=0}^{d} h_i(P)$, which is the total number of facets of $\Gamma(P)$, is even. $\square$

3.5.4 The Case $h_1(\partial\Gamma(P)) = 1$

A slightly more complicated argument allows us to extend the result of Proposition 3.13 to the case where $h_1(\partial\Gamma(P)) = 1$ and some higher $h_j(\partial(\Gamma(P))$ is zero.

Proposition 3.14. *Let P be a $(d-1)$-dimensional simplicial poset such that $\Gamma(P)$ is a ball, $h_1(\partial\Gamma(P)) = 1$, $h_j(\partial\Gamma(P)) = 0$ for some $1 < j < d-1$, and $h_k(P) = 0$ for some $0 < k < d$. Then $\sum_{i=0}^{d} h_i(P)$ is even.*

Proof. The assumption $h_1(\partial\Gamma(P)) = 1$ implies that $\partial\Gamma(P)$ has d vertices. Let $W = \{w_1, \ldots, w_d\}$ be the set of exterior vertices of $\Gamma(P)$. Let V be a set of d vertices of $\Gamma(P)$. If $V \neq W$ then V contains some interior vertex, so by Lemma 3.12 we know that there are an even number of facets of $\Gamma(P)$ with vertex set V. In particular, given any set S of $(d-1)$ vertices of $\Gamma(P)$ there are an even number (possibly zero) of facets with vertex set V that contain S.

If there are no facets with vertex set W then we are done, so assume F is a facet with vertex set W. Let W' be a set of $d-1$ distinct elements of W. Since $h_j(\partial\Gamma(P)) = 0$ we know by Theorem 3.2 that the number of boundary faces of $\Gamma(P)$ with vertex set W' is even. Because $\Gamma(P)$ is a manifold each interior face with vertex set W' is contained in two facets of $\Gamma(P)$ and each boundary face with vertex set W' is contained in one facet of $\Gamma(P)$. Therefore there are an even number of facets of $\Gamma(P)$ that contain the vertices

W'. As argued above there are an even number of facets on vertex sets other than W that contain W'. Thus in total there are an even number of facets on vertex set W. Hence we have an even total number of facets, which gives the desired result. $\square$

3.6 Constructions

We now turn our attention to constructing simplicial cell decompositions of balls with prescribed h-vectors. The balls that we construct are all shellable. We use the following result of Björner [5, Proposition 4.3] to prove that the complexes that we build are actually balls.

Proposition 3.15. *Let $\Gamma(P)$ be a shellable CW-complex of dimension $d-1$. If every $(d-2)$-cell is a face of at most two $(d-1)$-cells and some $(d-2)$-cell is a face of only one $(d-1)$-cell then $\Gamma(P)$ is homeomorphic to a $(d-1)$-ball.*

The first theorem of this section presents our basic construction method. The remainder of the section gives some extensions of this construction that allow us to obtain additional h-vectors.

Theorem 3.16. *Let $\mathbf{h} = (h_0, h_1, \ldots, h_{d-1}, h_d) \in \mathbb{N}^{d+1}$ with $h_0 = 1$ and $h_d = 0$. Let $\partial h_j = \sum_{i=0}^{j}(h_i - h_{d-i})$.*

1. *If $\partial h_j > 0$ for $0 \leq j \leq \lfloor (d-1)/2 \rfloor$ then there exists a poset P such that $\Gamma(P)$ is a $(d-1)$-ball and $h(P) = \mathbf{h}$.*
2. *Alternatively, let $0 < n < \lfloor (d-1)/2 \rfloor$ be the smallest number such that $\partial h_n = 0$. If $\sum_{i=0}^{d} h_i$ is even and $\partial h_l \leq \sum_{i=0}^{n-1} h_{l-i}$ for $n+1 \leq l \leq d-(n+1)$ then there exists a poset P such that $\Gamma(P)$ is a $(d-1)$-ball and $h(P) = \mathbf{h}$.*

Many of the conditions in Theorem 3.16 are related to the restrictions on the h-vectors of balls in Theorem 3.5. Also note that the inequalities in the second part of Theorem 3.16 are known to be necessary by Murai's result, Theorem 3.8.

Proof. We first present the notation that we will we use to describe the facets of our ball. We then recursively construct our ball by explaining how each facet of the ball is attached to the union of the previous facets. In Claim 1 we prove that this process results in a well-defined CW-complex. In Claim 2 we show that the complex is shellable with the desired restriction faces. Finally, in Claim 3 we show we have actually constructed a ball by proving that the conditions of Proposition 3.15 are satisfied.

We begin with the notation for the facets of our ball. The facets are denoted by F_i for $1 \leq i \leq \sum_{i=0}^{d} h_i$. Each facet F_i contains d vertices. We label these vertices $\{1\}_i, \{2\}_i, \ldots, \{d\}_i$. For any set $S \subseteq [d]$ we denote by $\{S\}_i$ the face of F_i containing exactly the vertices $\{\{l\}_i\}_{l \in S}$. For example $\{1, 2, 3, 4\}_3$ is the face of F_3 containing the vertices $\{1\}_3, \{2\}_3, \{3\}_3$, and $\{4\}_3$. We use the notation $\{a : b\}$ and $\{a : b\}_c$ to refer to $\{a, a+1, \ldots, b\}$ and $\{a, a+1, \ldots, b\}_c$, respectively.

Next we describe recursively how to construct the ball using the facets F_i. Let Δ_j be the complex $\cup_{i=1}^{j} F_i$. For each facet F_i we describe below the identifications of faces of F_i with faces in Δ_{i-1}. Most of the vertices of F_i will be identified with vertices of Δ_{i-1}, but in some cases F_i may contain a new vertex. For example, we may state that $\{1\}_2$ is identified with $\{1\}_1$ or that $\{1\}_2$ is a new face. In general, we choose the vertex labels such that two identified faces contain vertices labeled by the same numbers.

First we consider the case where all of the ∂h_i are strictly positive. Let $a = \sum_{i=0}^{d} h_i$, which is the total number of facets in our shelling. For $1 \leq k \leq a-1$ let c_k be the integer such that $\sum_{i=0}^{c_k-1} h_i < k+1 \leq \sum_{i=0}^{c_k} h_i$. Thus c_k measures the location where the sum of the entries of the vector $\mathbf{h}$ reaches $k+1$. Set $c_0 = 0$. Then $|\{k : c_k = j\}| = h_j$. As an example, if $\mathbf{h} = (1,2,0,0,1,0)$ then $a = 4$, $c_0 = 0$, $c_1 = 1$, $c_2 = 1$, and $c_3 = 4$.

We begin the shelling with the facet F_1 which cannot have any identifications with any previous facets, hence $|\sigma(F_1)| = 0$. The remaining facets are added in pairs F_i, F_{i+1} where i is even. The restriction faces of F_i and F_{i+1} are $\{1 : c_{i/2}\}_i$ and $\{c_{i/2}+1 : c_{i/2}+c_{a-i/2}\}_{i+1}$, respectively. We are pairing a facet contributing to the start of the h-vector with a facet contributing to the end of the h-vector and then working our way inward to the center of the h-vector with the subsequent pairs of facets. We stop after adding the facet F_a.

We now describe how the facets F_i and F_{i+1} are attached to our complex. For i even, we introduce a new face $\{1 : c_{i/2}\}_i$. Let $S \subseteq [d]$. If $S \supseteq \{1 : c_{i/2}\}$ then $\{S\}_i$ cannot be identified with any face in a previous facet. If $S \supseteq \{c_{i/2-1}+1 : c_{i/2}\}$ but $S \not\supseteq \{1 : c_{i/2-1}\}$ then identify $\{S\}_i$ with $\{S\}_{i-1}$. For all other sets $S \subseteq [d]$ identify $\{S\}_i$ with $\{S\}_1$. The fact that these identifications are well defined follows from the case $k = i$ of Claim 1 below.

Continuing the shelling, F_{i+1} is identified with F_i except we replace the face $\{c_{i/2}+1 : c_{i/2}+c_{a-i/2}\}_i$ by a new face $\{c_{i/2}+1 : c_{i/2}+c_{a-i/2}\}_{i+1}$ with the same boundary. The fact that all of the ∂h_i are positive ensures that $c_{i/2}+c_{a-i/2}$ never exceeds d, so the construction can proceed as described.

Claim 1. *Fix i even with $2 \leq i \leq a$ and k even with $2 \leq k \leq i$. Let $S \subseteq [d]$ such that $S \not\supseteq \{c_{i/2-1}+1 : c_{i/2}\}$ and $S \not\supseteq \{1 : c_{k/2-1}\}$. Then $\{S\}_{k-1} = \{S\}_1$.*

Proof of Claim 1. Our proof is by induction on i. The base case $i = 2$ is trivial. Assuming the result for $i = i'-2$ we prove it for $i = i'$. The inductive hypothesis allows us to assume that the construction of $\Delta_{i'-1}$ is well defined.

We prove the case $i = i'$ by induction on k. Again, the base case $k = 2$ is trivial. We assume the claim for $k = k' - 2$ and prove it for $k = k'$.

We first show that $\{S\}_{k'-1} = \{S\}_{k'-2}$. By the construction of the odd index facets, this follows from showing $S \not\supseteq \{c_{(k'-2)/2} + 1 : c_{(k'-2)/2} + c_{a-(k'-2)/2}\}$. By assumption $S \not\supseteq \{c_{i'/2-1} + 1 : c_{i'/2}\}$. Thus it is enough to show

$$c_{(k'-2)/2} + 1 \leq c_{i'/2-1} + 1 \qquad \text{and} \qquad c_{i'/2} \leq c_{(k'-2)/2} + c_{a-(k'-2)/2}. \tag{3.4}$$

The first inequality in (3.4) follows from the monotonicity of the c_l. Again using the monotonicity of the c_l and the fact that $i' \leq a$ we have

$$c_{i'/2} \leq c_{a-i'/2} \leq c_{a-(k'-2)/2} \leq c_{(k'-2)/2} + c_{a-(k'-2)/2},$$

proving the second inequality.

We complete the proof of Claim 1 by showing $\{S\}_{k'-2} = \{S\}_1$. By assumption, $S \not\supseteq \{1 : c_{k'/2-1}\} = \{1 : c_{(k'-2)/2}\}$. Thus, if $S \supseteq \{c_{(k'-2)/2-1} + 1 : c_{(k'-2)/2}\}$ then $S \not\supseteq \{1 : c_{(k'-2)/2-1}\}$. In this case, by our construction of the even index facets we have $\{S\}_{k'-2} = \{S\}_{k'-3}$ and by the inductive hypothesis $\{S\}_{k'-3} = \{S\}_1$, giving the desired result. If $S \not\supseteq \{c_{(k'-2)/2-1} + 1 : c_{(k'-2)/2}\}$ then our construction identifies $\{S\}_{k'-2}$ and $\{S\}_1$, completing the proof. $\square$

Let $\{\hat{j}\}_k$ be the codimension-one face of F_k that does not contain the vertex $\{j\}_k$.

Claim 2. *Let $2 \leq k \leq a$. Then*

$$\begin{aligned} &\textit{for } k = i \textit{ even} && F_i \cap \Delta_{i-1} = \cup_{j=1}^{c_{i/2}} \{\hat{j}\}_i \\ &\textit{and for } k = i+1 \textit{ odd} && F_{i+1} \cap \Delta_i = \cup_{j=c_{i/2}+1}^{c_{i/2}+c_{a-i/2}} \{\hat{j}\}_{i+1}. \end{aligned}$$

Hence the F_i form a shelling order with $|\sigma(F_i)| = c_{i/2}$ and $|\sigma(F_{i+1})| = c_{a-i/2}$.

Proof of Claim 2. For $1 \leq j \leq c_{i/2-1}$ each of the faces $\{\hat{\jmath}\}_i$ is also in F_{i-1} while for $c_{i/2-1} < j \leq c_{i/2}$ the face $\{\hat{\jmath}\}_i$ is also a face of F_1. Therefore $\cup_{j=1}^{c_{i/2}}\{\hat{\jmath}\}_i \subseteq (F_i \cap \Delta_{i-1})$. To see the reverse inclusion note that any face in $F_i \backslash \left(\cup_{j=1}^{c_{i/2}}\{\hat{\jmath}\}_i\right)$ contains the face $\{1 : c_{i/2}\}_i$, which is a new face and therefore not in Δ_{i-1}.

The proof for F_{i+1} is handled in a similar manner with $\{\hat{\jmath}\}_{i+1} \subseteq F_i$ for $c_{i/2} + 1 \leq j \leq c_{i/2} + c_{a-i/2}$ and $\{c_{i/2} + 1 : c_{i/2} + c_{a-i/2}\}_{i+1} \notin \Delta_i$. The last part of Claim 2 now follows from the definition of a shelling. $\square$

Claim 3. *For $1 \leq p \leq a$ each codimension-one face of Δ_p is contained in at most two facets and there exists a codimension-one face of Δ_p that is contained in only one facet.*

Proof of Claim 3. We first show that each codimension-one face is contained in at most two facets. We do this by showing that any codimension-one face in $F_l \cap \Delta_{l-1}$ for $1 < l \leq a$ is either a face of $F_{l-1} \backslash \Delta_{l-2}$ or a face of $F_1 \backslash (\cup_{i=2}^{l-1} F_i)$.

Let $l > 0$ be even and consider $F_l \cap \Delta_{l-1}$. By Claim 2 we need to consider the faces $\{\hat{\jmath}\}_l$ for $1 \leq j \leq c_{l/2}$. Using the attachment rules given before Claim 1, for $1 \leq j \leq c_{l/2-1}$ we see that $\{\hat{\jmath}\}_l$ is a face in $F_{l-1} \backslash \Delta_{l-2}$, while for $c_{l/2-1} + 1 \leq j \leq c_{l/2}$ the face $\{\hat{\jmath}\}_l$ is in $F_1 \backslash (\cup_{i=2}^{l-1} F_i)$.

Now consider the case $F_{l+1} \cap \Delta_l$. By Claim 2 we are interested in the faces $\{\hat{\jmath}\}_{l+1}$ for $c_{l/2} + 1 \leq j \leq c_{l/2} + c_{a-l/2}$. Again using the attachment rules given before Claim 1, all of these are faces in $F_l \backslash \Delta_{l-1}$, as desired.

Using Claim 2, the faces $\{\hat{\jmath}\}_1$ for $j > c_{\lfloor a/2 \rfloor}$ never appear in any F_l with $l > 1$. Since $h_d = 0$ we know $c_{\lfloor a/2 \rfloor} < d$, so the codimension-one face $\{\hat{d}\}_1$ is only contained in the facet F_1, completing the proof of Claim 3. $\square$

By Claim 3 and Proposition 3.15 we know that Δ_a is a ball. Using the $|\sigma(F_i)|$ from Claim 2 to count the contribution of each facet to the h-vector along with the fact $|\{i : c_i = j\}| = h_j$ we know that Δ_a has the desired h-vector.

We now consider the case where $0 < n < \lfloor (d-1)/2 \rfloor$ is the smallest integer such that $\partial h_n = 0$. Define a new vector $\mathbf{h}' = (h'_0, h'_1, \ldots, h'_{d-1}, h'_d)$ by $h'_i = h_i$ for $i \neq d-n$ and $h'_{d-n} = h_{d-n} - 1$. From the definition of the ∂h_i, in order for $\partial h_{n-1} > 0$ and $\partial h_n = 0$ we must have $h_{d-n} > 0$, so the vector $\mathbf{h}'$ has non-negative entries. Additionally, for $n \leq i \leq \lfloor (d-1)/2 \rfloor$ we have $\partial h'_i = \partial h_i + 1$ ensuring that all of the $\partial h'_i$ are strictly positive. We can therefore apply the construction of the previous case to create a ball with h-vector $\mathbf{h}'$. In what follows we take $a = \sum_{i=0}^{d} h'_i = (\sum_{i=0}^{d} h_i) - 1$ to match the definition of a in the previous case. Similarly, c_k for $0 \leq k \leq a-1$ measures the location where the sum of the entries of $\mathbf{h}'$ reaches $k+1$.

By assumption $\sum_{i=0}^{d} h_i$ is even, hence the construction of the ball with h-vector $\mathbf{h}'$ ends with the facet F_a with a odd. We complete the construction of a ball with h-vector $\mathbf{h}$ by adding a facet F_{a+1}. We attach F_{a+1} to the ball Δ_a using the rules given before Claim 1 for attaching an even index facet but acting as if $\tilde{c}_{(a+1)/2} = d-n$, so $\{1 : d-n\}_{a+1}$ is a new face (when referring to the facet F_{a+1} we use the notation $\tilde{c}_{(a+1)/2} = d-n$ to avoid confusion with $c_{(a+1)/2}$ as defined from the vector $\mathbf{h}'$). To complete the proof we must extend the results of Claims 1, 2, and 3 to include the additional facet F_{a+1}.

First we prove Claim 1 for the case $i = a+1$. The proof is the same as for smaller i values except that the second inequality in (3.4) requires a different justification. Rewriting this inequality, for $1 \leq j \leq (a-1)/2$ we must show

$$d - n \leq c_j + c_{a-j}.$$

First consider the case $c_j \geq d-2n$. Since $n < \lfloor (d-1)/2 \rfloor$, adding the equations $\sum_{m=0}^{d} h_m = a+1$ and $\sum_{m=0}^{n}(h_m - h_{d-m}) = 0$ and then removing some non-negative terms from the left-hand side yields $\sum_{m=0}^{n} h_m \leq (a+1)/2$. Hence $\sum_{m=0}^{n} h'_m \leq (a+1)/2$ and $c_{(a+1)/2} \geq n$. Since $a-j \geq (a+1)/2$ we have $c_{a-j} \geq c_{(a+1)/2} \geq n$, completing the proof of this case.

For the case $1 \leq c_j \leq d-2n-1$ note that we can rewrite the assumption $\partial h_l \leq \sum_{m=0}^{n-1} h_{l-m}$ as

$$\sum_{m=0}^{l-n} h_m \leq \sum_{m=0}^{l} h_{d-m} \qquad \text{or} \qquad \sum_{m=1}^{l-n} h'_m \leq \sum_{m=0}^{l} h'_{d-m}$$

where $n+1 \leq l \leq d-(n+1)$. By the second inequality, choosing l such that $c_j = l-n$ we have $c_{a-j} \geq d-l$. Therefore $c_j + c_{a-j} \geq d-n$, as desired.

We extend Claim 2 by showing

$$F_{a+1} \cap \Delta_a = \cup_{j=1}^{d-n} \{\hat{\jmath}\}_{a+1}. \tag{3.5}$$

This follows from the proof of the even case of Claim 2 by treating $\tilde{c}_{(a+1)/2} = d-n$.

To allow $p = a+1$ in Claim 3 we consider the codimension-one faces in $F_{a+1} \cap \Delta_a$. By Equation (3.5) these are $\{\hat{\jmath}\}_{a+1}$ for $1 \leq j \leq d-n$. Using the attachment rules for the F_i, for $1 \leq j \leq c_{(a-1)/2}$ we see that $\{\hat{\jmath}\}_{a+1}$ is a face in $F_a \backslash \Delta_{a-1}$, while for $c_{(a-1)/2}+1 \leq j \leq d-n$ the face $\{\hat{\jmath}\}_{a+1}$ is in $F_1 \backslash (\cup_{i=2}^{a} F_i)$.

The faces $\{\hat{\jmath}\}_1$ for $j > d-n$ never appear in any F_l with $l > 1$. Since $n > 0$ the codimension-one face $\{\hat{d}\}_1$ is only contained in the facet F_1, completing the proof of the extended version of Claim 3. $\square$

We next present a slight augmentation of the previous theorem that allows us to deal with some additional cases involving h-vectors that have a single sequence of non-zero entries.

Theorem 3.17. *Let $\mathbf{h} = (h_0, h_1, \ldots, h_{d-1}, h_d) \in \mathbb{N}^{d+1}$ with $h_0 = 1$. Assume that there exists $k \in \{1, 2, \ldots, d-1\}$ such that $h_j = 0$ for $j > k$ and $h_j > 0$ for $1 \leq j \leq k$. Define $\mathbf{h}' = (1, h_1 - 1, h_2 - 1, \ldots, h_k - 1, 0, \ldots, 0)$. If $\mathbf{h}'$ satisfies the conditions of Theorem 3.16 then there exists a poset P such that $\Gamma(P)$ is a $(d-1)$-ball and $h(P) = \mathbf{h}$.*

Proof. We once again construct a shellable CW-complex with the desired h-vector. We begin by building a *simplicial complex* on vertex set $[d+1]$. We think of the faces of this simplicial complex as subsets of $[d+1]$ as well as topological simplexes.

For $0 \leq i \leq k$ define $G_i = [d+1] - \{i+1\}$, a face of our simplicial complex. Let Δ_j be the simplicial complex whose facets are $\{G_i\}_{i=0}^{j}$. Then for $1 \leq i \leq k$

$$G_i \cap \Delta_{i-1} = \cup_{j=1}^{i} \left(G_i - \{j\} \right).$$

Hence $G_1, \ldots, G_k$ is a shelling order for Δ_k with $\sigma(G_i) = \{1, 2, \ldots, i\}$ and $|\sigma(G_i)| = i$.

We complete the proof by performing the construction of Theorem 3.16 on the vector $\mathbf{h}'$ with a few alterations. We omit the initial facet F_1 in Theorem 3.16. Instead we attach all of our additional facets to the boundary of Δ_k. In our new construction we replace the vertices $\{j\}_1$, $1 \leq j \leq d$, from Theorem 3.16 with the vertices $[d+1] \backslash \{k+2\}$ of Δ_k, identifying vertices in the order preserving way. We then replace the faces of ∂F_1 from Theorem 3.16 with the faces defined by the corresponding sets of vertices of Δ_k.

We claim that performing the construction of Theorem 3.16 with this alteration gives a shellable ball, with a shelling order given by concatenating the order $G_1, G_2, \ldots, G_k$ with the order given by Theorem 3.16. To prove this we take every facet of ∂F_1 that is contained in a later facet in the construction of Theorem 3.16 and show that the corresponding $(d-1)$-subset of $[d+1] \backslash \{k+2\}$ is a face of $\partial \Delta_k$. Let $H = [d+1] \backslash \{j, k+2\}$ be a $(d-1)$-subset of $[d+1] \backslash \{k+2\}$. For $1 \leq j \leq k+1$, the only facet of Δ_k that contains H is G_{j-1}, so H is in $\partial\Delta_k$. For $k+3 \leq j \leq d+1$, the facet of ∂F_1 corresponding to H will never be used in the construction of Theorem 3.16 since $h_l = 0$ for $l > k$.

Totaling the contributions of all of the $|\sigma(G_i)|$ shows that the ball created in this manner has the desired h-vector. □

Note that when $d-1$ is odd, by taking $k = d-1$ in the previous theorem it is possible to create a ball with an odd number of facets and a zero in its boundary sphere. Hence in the odd dimension case, there are some situations where a zero in the boundary sphere forces and even number of facets (these are the results of Section 3.5) and some situations where there is a zero in the boundary sphere h-vector but an odd number of facets. This is in contrast to the even dimension case where a zero in the boundary sphere h-vector always forces an even number of facets.

We conclude this section with a construction specific to dimension five. This construction provides more examples of balls with a zero in the boundary sphere h-vector and an odd number of facets. This result allows us to complete the characterization of the possible h-vectors of five-balls in the following section.

Proposition 3.18. *Let* $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4, 0, 0) \in \mathbb{N}^7$ *with* $h_0 = 1$, $h_1, h_2 \neq 0$, $\sum_{i=0}^{4} h_i$ *odd, and* $\partial h_2 = 1 + h_1 + h_2 - h_4 = 0$. *Then there exists a poset* P *such that* $\Gamma(P)$ *is a five-ball and* $h(P) = \mathbf{h}$.

Proof. We begin this construction by using Theorem 3.16 to create a 5-ball with h-vector $(1, 0, 0, h_3 - 1, 0, 0, 0)$. Note that since $1 + h_1 + h_2 - h_4 = 0$, the parity of h_3 is the same as the parity of $\sum_{i=0}^{4} h_i$, which we assumed to be odd. Therefore $h_3 - 1$ is even and non-negative.

When we complete this construction the facets of the boundary of our ball are

$$\{1,2,3,4,5\}_1, \qquad \{1,2,3,4,6\}_1, \qquad \{1,2,3,5,6\}_1,$$

$$\{2,3,4,5,6\}_{h_3}, \qquad \{1,3,4,5,6\}_{h_3}, \qquad \text{and} \qquad \{1,2,4,5,6\}_{h_3}.$$

The fact that the first three faces are on the boundary follows from the discussion at the end of Claim 3 in the proof of Theorem 3.16. The last three faces are on the boundary because F_{h_3} is the last facet added to our ball and $\{4,5,6\}_{h_3}$ is a new face when F_{h_3} is added in the construction of Theorem 3.16.

Further, all of the faces of F_{h_3} are identified with the corresponding faces of F_1 except for $\{1,2,3\}_{h_3} = \{1,2,3\}_{h_3-1}$, $\{4,5,6\}_{h_3}$, and all faces containing one of these two faces (there is also the easier case where $h_3 = 1$ and we have only one facet in this initial part of the shelling).

We now describe the next six facets of our shelling, altering our notation slightly to make the description easier to follow.

$F_{h_3+1} = \{1,2,3,4,5,7\}_{h_3+1}$ where $\{7\}_{h_3+1}$ is a new vertex.
$\{1,2,3,4,5\}_{h_3+1}$ is identified with $\{1,2,3,4,5\}_1$.
Hence $\sigma(F_{h_3+1}) = \{7\}_{h_3+1}$.

$F_{h_3+2} = \{1,2,3,4,6,7\}_{h_3+2}$.
$\{1,2,3,4,6\}_{h_3+2}$ is identified with $\{1,2,3,4,6\}_1$.
$\{1,2,3,4,7\}_{h_3+2}$ is identified with $\{1,2,3,4,7\}_{h_3+1}$.
Hence $\sigma(F_{h_3+2}) = \{6,7\}_{h_3+2}$.

$F_{h_3+3} = \{1,2,3,5,6,7\}_{h_3+3}$.
$\{1,2,3,5,6\}_{h_3+3}$ is identified with $\{1,2,3,5,6\}_1$.
$\{1,2,3,5,7\}_{h_3+3}$ is identified with $\{1,2,3,5,7\}_{h_3+1}$.

$\{1,2,3,6,7\}_{h_3+3}$ is identified with $\{1,2,3,6,7\}_{h_3+2}$.

Hence $\sigma(F_{h_3+3}) = \{5,6,7\}_{h_3+3}$.

$F_{h_3+4} = \{1,2,4,5,6,7\}_{h_3+4}$ with $\{4,5,6\}_{h_3+4} = \{4,5,6\}_{h_3}$.

$\{1,2,4,5,6\}_{h_3+4}$ is identified with $\{1,2,4,5,6\}_{h_3}$.

$\{1,2,4,5,7\}_{h_3+4}$ is identified with $\{1,2,4,5,7\}_{h_3+1}$.

$\{1,2,4,6,7\}_{h_3+4}$ is identified with $\{1,2,4,6,7\}_{h_3+2}$.

$\{1,2,5,6,7\}_{h_3+4}$ is identified with $\{1,2,5,6,7\}_{h_3+3}$.

Hence $\sigma(F_{h_3+4}) = \{4,5,6,7\}_{h_3+4}$.

$F_{h_3+5} = \{1,3,4,5,6,7\}_{h_3+5}$ with the identification $\{4,5,6\}_{h_3+5} = \{4,5,6\}_{h_3}$ and the new face $\{4,5,6,7\}_{h_3+5}$.

$\{1,3,4,5,6\}_{h_3+5}$ is identified with $\{1,3,4,5,6\}_{h_3}$.

$\{1,3,4,5,7\}_{h_3+5}$ is identified with $\{1,3,4,5,7\}_{h_3+1}$.

$\{1,3,4,6,7\}_{h_3+5}$ is identified with $\{1,3,4,6,7\}_{h_3+2}$.

$\{1,3,5,6,7\}_{h_3+5}$ is identified with $\{1,3,5,6,7\}_{h_3+3}$.

Hence $\sigma(F_{h_3+5}) = \{4,5,6,7\}_{h_3+5}$.

$F_{h_3+6} = \{2,3,4,5,6,7\}_{h_3+6}$ with the identification $\{4,5,6\}_{h_3+6} = \{4,5,6\}_{h_3}$ and the new face $\{4,5,6,7\}_{h_3+6}$.

$\{2,3,4,5,6\}_{h_3+6}$ is identified with $\{2,3,4,5,6\}_{h_3}$.

$\{2,3,4,5,7\}_{h_3+6}$ is identified with $\{2,3,4,5,7\}_{h_3+1}$.

$\{2,3,4,6,7\}_{h_3+6}$ is identified with $\{2,3,4,6,7\}_{h_3+2}$.

$\{2,3,5,6,7\}_{h_3+6}$ is identified with $\{2,3,5,6,7\}_{h_3+3}$.

Hence $\sigma(F_{h_3+6}) = \{4,5,6,7\}_{h_3+6}$.

Examining the $|\sigma(F_i)|$, the ball we have constructed has h-vector $(1,1,1,h_3,3,0,0)$.

We finish building our ball using a slightly altered version of the construction of Theorem 3.16 on the vector $(1, h_1 - 1, h_2 - 1, 0, h_4 - 3, 0, 0)$. In place of the initial facet from Theorem 3.16 we use the final facet F_{h_3+6} from the above construction (with the order preserving identification of the two facets' vertices).

Consider the construction of Theorem 3.16 for the h-vector $(1, h_1 - 1, h_2 - 1, 0, h_4 - 3, 0, 0)$. We have $c_{(a-1)/2} \leq 2$, hence by the proof of Claim 3 the only codimension-one faces of F_1 that are attached to facets F_i with $i > 1$ during the construction are $\{\hat{1}\}_1$ and $\{\hat{2}\}_1$.

When we replace the initial facet in Theorem 3.16 with F_{h_3+6} from the above construction, the corresponding codimension-one faces that will be attached to later facets are $\{3, 4, 5, 6, 7\}_{h_3+6}$ and $\{2, 4, 5, 6, 7\}_{h_3+6}$. These two faces are both on the boundary of our above constructed ball. Thus we can finish the shelling in this manner, creating a ball with the desired h-vector. $\square$

3.7 A Summary of Known Conditions

Using the results of the previous two sections we now fully characterize the h-vectors of simplicial posets that are balls in all even dimensions as well as dimensions three and five.

Theorem 3.19 (Even Dimensions). *Let d be an odd integer and $\mathbf{h} = (h_0, h_1, \ldots, h_{d-1}, h_d) \in \mathbb{Z}^{d+1}$ with $h_0 = 1$ and $h_d = 0$. Define $\partial h_j = \sum_{i=0}^{j}(h_i - h_{d-i})$ for $0 \leq j \leq d-1$. Then there exists a simplicial poset P such that $\Gamma(P)$ is a $(d-1)$-ball and $\mathbf{h} = h(P)$ if and only if the following all hold.*

1. *$h_i \geq 0$ for $1 \leq i \leq d-1$.*

2. $\partial h_i \geq 0$ *for* $0 \leq i \leq (d-1)/2$.

3. *If* $\partial h_i = 0$ *for any* $1 \leq i \leq (d-1)/2$ *then* $\sum_{j=0}^{d-1} h_j$ *is even.*

4. *If* $\partial h_i = 0$ *for any* $1 \leq i \leq (d-1)/2$ *then* $\partial h_l \leq \sum_{j=0}^{i-1} h_{l-j}$ *for* $i+1 \leq l \leq d-(i+1)$.

Proof. For the first three conditions necessity follows directly from Theorem 3.5, while for the last condition it is a result of Theorem 3.8. When the second condition is satisfied with all strict inequalities sufficiency is proved using the first case of Theorem 3.16; otherwise we use the second case of Theorem 3.16. □

Proposition 3.20 (Dimension 3)**.** *Let* $\mathbf{h} = (1, h_1, h_2, h_3, 0) \in \mathbb{Z}^5$. *Then there exists a simplicial poset* P *such that* $\Gamma(P)$ *is a three-ball and* $\mathbf{h} = h(P)$ *if and only if the following all hold.*

1. $h_i \geq 0$ *for* $1 \leq i \leq 3$.

2. $h_3 \leq h_1 + 1$.

3. *If* $h_1 = 0$ *and* $h_3 = 1$ *then* h_2 *is even.*

Proof. The necessity of the first two conditions follows directly from Theorem 3.5, while the third condition is a consequence of Proposition 3.10.

When $h_3 < h_1 + 1$ sufficiency follows from the first case of Theorem 3.16. If $h_3 = h_1 + 1$ and h_2 is even the second case of Theorem 3.16 gives the desired result. Otherwise, $h_3 = h_1 + 1 > 1$ and h_2 is odd which means all of the h_i for $0 \leq i \leq 3$ are non-zero and we can apply Theorem 3.17 to obtain the needed construction. □

Proposition 3.21 (Dimension 5)**.** *Let* $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4, h_5, h_6) \in \mathbb{Z}^7$ *with* $h_0 = 1$ *and* $h_6 = 0$. *Let* $\partial h_j = \sum_{i=0}^{j}(h_i - h_{6-i})$ *for* $0 \leq j \leq 5$. *Then there exists a simplicial poset* P *such that* $\Gamma(P)$ *is a five-ball and* $\mathbf{h} = h(P)$ *if and only if the following all hold.*

1. $h_i \geq 0$ *for* $1 \leq i \leq 5$.

2. $\partial h_1 \geq 0$. *If* $\partial h_1 = 0$ *then* $h_i \geq \partial h_i$ *for* $0 \leq i \leq 5$. *If* $\partial h_1 = 0$ *and* $h_j = 0$ *for some* $1 \leq j \leq 4$ *then* $\sum_{i=0}^{5} h_i$ *is even.*

3. $\partial h_2 \geq 0$. *If* $\partial h_2 = 0$ *and* $h_1 = 0$ *or* $h_2 = 0$ *then* $\sum_{i=0}^{5} h_i$ *is even.*

Proof. Then necessity of condition one and the first inequality in the other two conditions follow from Theorem 3.5. The additional inequalities when $\partial h_1 = 0$ come from Theorem 3.7. The remainder of condition two comes from Proposition 3.13. For the third condition, the case $h_1 = 0$ follows from Proposition 3.10 while the case $h_2 = 0$ is a result of Proposition 3.11.

For sufficiency, if there are strict inequalities in the second and third conditions we use the first case of Theorem 3.16. If $\partial h_1 = 0$ and $\sum_{i=0}^{5} h_i$ is even then we apply the second case of Theorem 3.16. If $\partial h_1 = 0$ and $\sum_{i=0}^{5} h_i$ is odd then all of the h_i for $1 \leq i \leq 5$ are non-zero, so we apply Theorem 3.17. It is not hard to check that reducing the elements of the h-vector by one will preserve the inequalities $h_i \geq \partial h_i$ for $0 \leq i \leq 5$. The case $h_3 \geq \partial h_3$ uses the fact that the sum of the h_i is odd.

If $\partial h_1 > 0$ and $\partial h_2 = 0$, the inequality in the second case of Theorem 3.16 is always trivially satisfied. This gives the needed construction whenever $\sum_{i=0}^{5} h_i$ is even. For the case where $\sum_{i=0}^{5} h_i$ is odd, note that h_3 must be odd and hence non-zero. Therefore $h_i > 0$ for $1 \leq i \leq 4$. When $h_5 > 0$ we can apply Theorem 3.17 to obtain the desired construction; otherwise we use Proposition 3.18. $\square$

For even-dimensional balls (d odd), when any entry of the boundary h-vector is zero the sum of the h_i of the ball must have even parity. In the odd-dimensional case, this relationship is lost and more subtle behavior occurs. In particular, whether or not some of the

h_i of the ball are zero needs to be considered, resulting in some of the more complicated conditions and the extra construction in the dimension five case. In his recent preprint, Murai [26, Theorem 1.2] gives a complete description of the possible h-vectors of odd-dimensional balls. His additional necessary conditions all describe situations in which the number of facets of the ball must be even. These conditions involve inequalities on the entries of the h-vectors of the ball and its boundary sphere in addition to the existence of zero entries in these vectors. The proof of the necessity of these conditions follows from additional arguments involving the face ring. In particular, Proposition 3.14 above is a special case of a more general condition involving a zero in the ball's h-vector, a zero in the boundary sphere h-vector, and a separate entry where $\partial h_i \leq i$, which together force the ball to have an even number of facets. To prove sufficiency, Murai extends the basic idea of the constructions of Section 3.6 to obtain additional h-vectors.

Chapter 4

f-vectors of Products and Surgery

We now return to the setting of simplicial complexes - in particular triangulated manifolds. Our goal is to understand how surgeries (see section 1.5 for an introduction) alter the face numbers of a manifold. We use our results on surgeries to study the relationship between the face numbers and Betti numbers of manifolds.

The structure of the chapter is as follows. In Section 4.1 we discuss a method for triangulating the Cartesian product of two simplicial complexes and we derive a formula for the f-vector of this triangulation of the product. Transferring this to a useful formula in terms of h-vectors is in general difficult. In Section 4.2 we calculate h-vector formulas in the special case where one of the simplicial complexes in our product is a simplex or the boundary of a simplex. In Section 4.3 we apply these formulas to calculate the effects of surgery on the g-vector. Section 4.4 introduces the idea of the h''-vector of a homology manifold in order to discuss Kalai's conjectured lower bounds on the g-vector in terms of the Betti numbers. In cases where these bounds are known to hold, we combine them with the formulas of the previous section to describe new restrictions on the g-vectors of manifolds that admit certain types of surgeries. Finally, in Section 4.5 we present a collection of binomial coefficient relationships that are used throughout the chapter.

4.1 Cartesian Products of Simplicial Complexes

We begin by describing how we triangulate the Cartesian product $K \times L$ of two simplicial complexes. First order the vertices of K and L (in general the triangulation depends on this choice). The vertices of $K \times L$ are the pairs (v, w) where v is a vertex of K and w is a vertex of L. A set of pairs $\{(v_0, w_0), (v_1, w_1), \ldots, (v_a, w_a)\}$ is a face of $K \times L$ if and only if

- $v_0 \leq v_1 \leq \cdots \leq v_a$ in the ordering of the vertices of K,
- $w_0 \leq w_1 \leq \cdots \leq w_a$ in the ordering of the vertices of L.
- $\{v_0, v_1, \ldots, v_a\}$ is a face of K (note that the v_i need not be distinct), and
- $\{w_0, w_1, \ldots, w_a\}$ is a face of L.

Using this definition we derive a formula for the face numbers of $K \times L$. Let $F = \{(v_0, w_0), (v_1, w_1), \ldots, (v_a, w_a)\}$ be an a-face of $K \times L$. Let $H_1 = \{v_0, v_1, \ldots, v_a\}$ and $H_2 = \{w_0, w_1, \ldots, w_a\}$. Let $n = \dim H_1$ and $m = \dim H_2$. For each $i \in \{1, \ldots, a\}$ we have one of the following three cases

1. $v_i > v_{i-1}$ and $w_i = w_{i-1}$,
2. $v_i = v_{i-1}$ and $w_i > w_{i-1}$, or
3. $v_i > v_{i-1}$ and $w_i > w_{i-1}$.

In order for H_1 and H_2 to have the correct dimensions, cases one and three must occur a combined n times while cases two and three must happen a combined m times. Therefore, case three occurs $n + m - a$ times. This means case one occurs $a - m$ times while case two occurs $a - n$ times. Hence we have a total of $\binom{a}{a-m, a-n, n+m-a}$ a-faces of $K \times L$ using exactly the vertices of H_1 from K and the vertices of H_2 from L.

This formula allows us to calculate the total number of a-faces of $K \times L$ by summing the contributions of all pairs of faces of K and L. For $a \geq 0$ we have

$$f_a(K \times L) = \sum_{l=0}^{a} \sum_{i=0}^{a} f_i(K) f_l(L) \binom{a}{a-l, a-i, i+l-a}. \tag{4.1}$$

When $i+l<a$, no a-face of $K \times L$ is possible and we interpret the multinomial coefficient as zero. Note that the face numbers do not depend on the orderings of the vertices of K and L.

4.2 Formulas for the h-vectors of Products

Equation (4.1) gives the f-vector of a product $A^j \times C^k$ in terms of the f-vectors of A^j and C^k. The goal of this section is a similar formula in terms of the h-vectors of the product and the original complexes. Our main result is Theorem 4.2, which gives a useful formula for the h-vector of the product of a simplicial complex with either a simplex or the boundary of a simplex. Before we prove this result, we first calculate the change in the h-vector of a product $A^j \times C^k$ when a single k-face is added to C^k.

Proposition 4.1. *Let A^j and C^k be simplicial complexes of dimension j and k respectively and let $d=j+k$. If we add a single dimension k face to C^k then the change in the h-vector of the product of A^j and C^k is given by*

$$\delta h_b = \sum_{m=0}^{j+1} \left[\binom{d+1-m}{d+1-b} \binom{m-1}{b-k-1} \right] h_m(A^j). \tag{4.2}$$

If C^k has dimension $k-1 \geq 0$, so that we are adding in the first face of dimension k, equation (4.2) is still valid if the h-vector of the original product is replaced by the g-vector.

The last statement of the proposition is equivalent to calculating the h-vector of the original product as if the product had dimension $j+k-1$, even when its dimension is only $j+k-2$.

Proof. By formula (4.1), we know that we can write $\delta h_b = \sum_{m=0}^{j+1} c_{m,b} h_m(A^j)$ for some constants $c_{m,b}$ that depend on j and k. To calculate these coefficients we add a single facet to A^j such that the facet has a unique minimal new face with m vertices. The only change in the h-vector of A^j when we add this facet is that $h_m(A^j)$ increases by one. Therefore, by calculating the change in the h-vector of the product when this facet is added we can determine $c_{m,b}$.

Since each new face of A^j must contain the m vertices of the minimal new face, the changes in the face numbers of A^j are given by

$$\Delta f_i(A^j) = \binom{(j+1)-m}{(i+1)-m}.$$

Note that $\Delta f_{-1} = 0$. Applying equation (4.1), for $a \geq 0$ the number of a-faces in the product that result from combining new faces of A^j and the new k-face of C^k is

$$\Delta f_a = \sum_{i=0}^{a} \binom{j+1-m}{i+1-m} \binom{a}{a-k, a-i, i+k-a}.$$

Using Lemma 4.7, for $a \geq 0$ we have

$$\Delta f_a = \binom{a}{a-k} \binom{j+1-m+k}{a-m+1} = \binom{a}{k} \binom{d+1-m}{d-a}.$$

We use this formula to calculate the corresponding change in the h-vector of the product.

$$\begin{aligned}\Delta h_b &= \sum_{a=0}^{b} (-1)^{b-a} \binom{d-a+1}{d-b+1} \Delta f_{a-1} \\ &= \sum_{a=1}^{b} (-1)^{b-a} \binom{d-a+1}{d-b+1} \binom{a-1}{k} \binom{d+1-m}{d+1-a}.\end{aligned}$$

Because of the last binomial coefficient, the terms in the sum with $a < m$ are zero. Hence the desired result follows when $b < m$. We therefore assume that $b \geq m$. Examining the second binomial coefficient we see that it is sufficient to start the sum at $a = k+1$. With these simplifications and re-indexing we have

$$\Delta h_b = \sum_{a=k}^{b-1} (-1)^{b-a-1} \binom{d-a}{d-b+1} \binom{a}{k} \binom{d+1-m}{d-a}.$$

Noting that the product of the first and third binomial coefficients can be written as a multinomial coefficient we get further simplifications.

$$\begin{aligned}
\Delta h_b &= \sum_{a=k}^{b-1} (-1)^{b-a-1} \binom{a}{k} \binom{d+1-m}{d-b+1, b-1-a, a-m+1} \\
&= \sum_{a=k}^{b-1} (-1)^{b-a-1} \binom{a}{k} \binom{d+1-m}{d-b+1} \binom{b-m}{b-1-a} \\
&= (-1)^{b-1} \binom{d+1-m}{d-b+1} \sum_{a=k}^{b-1} (-1)^{a} \binom{a}{k} \binom{b-m}{a+1-m}.
\end{aligned}$$

In the case where $b-1 < k$, the desired equality follows because Δh_b is equal to an empty sum and $\delta h_b = 0$ by equation (4.2). If $b-1 \geq k$, by Lemma 4.8 we have

$$\begin{aligned}
\Delta h_b &= (-1)^{b-1} \binom{d+1-m}{d-b+1} (-1)^{(b-m)+(m-1)} \binom{m-1}{k-(b-m)} \\
&= \binom{d+1-m}{d+1-b} \binom{m-1}{b-k-1},
\end{aligned}$$

as desired. Note that to apply Lemma 4.8 we use the fact $b \geq m$. □

Theorem 4.2. *Let A^j be a simplicial complex of dimension j and let $d = j+k$. Then for $0 \leq b \leq d$*

$$g_b(A^j \times \partial\Delta^k) = \sum_{m=0}^{j+1} \left[\binom{j-m}{b-m} \binom{k+m}{b} - \binom{d+1-m}{d+1-b} \binom{m-1}{b-k-1} \right] h_m(A^j) \tag{4.3}$$

and for $0 \leq b \leq d+1$

$$h_b(A^j \times \Delta^k) = \sum_{m=0}^{j+1} \left[\binom{j-m}{b-m} \binom{k+m}{b} \right] h_m(A^j). \tag{4.4}$$

Proof. By Proposition 4.1, Equation (4.3) follows from Equation (4.4), so it suffices to prove the second relation. As in the previous proof, we know that there exist coefficients for the $h_m(A^j)$ such that the relation holds. To determine these coefficients we use the case where the facets of A^j are a subset of the facets of $\partial\Delta^{j+1}$. We then examine the change in the h-vector when we add an additional facet to this subset. The case $m = j+1$ requires a separate argument.

In [2], Billera, Cushman, and Sanders (BCS) describe a shelling order for products of simplices. We use this shelling order, applied multiple times, to shell the product of Δ^k and our subcomplex A^j of $\partial\Delta^{j+1}$.

Think of the vertices of $\partial\Delta^{j+1} \times \Delta^k$ as a rectangular lattice with $j+2$ rows (corresponding to the vertices of $\partial\Delta^{j+1}$) and $k+1$ columns (corresponding to the vertices of Δ^k). Each facet of $\partial\Delta^{j+1}$ corresponds to omitting one of these rows. We order these facets such that F_i for $0 \leq i \leq j+1$ is the facet where the $(i+1)$st row is removed (where row one is the top row). In our shelling order we shell $F_0 \times \Delta^k$, then $F_1 \times \Delta^k$, and so on.

The facets of each $F_i \times \Delta^k$ are shelled using the BCS ordering, which we now describe. If we disregard the row that does not contain a vertex of F_i, each facet of $F_i \times \Delta^k$ corresponds to a path of up and rightward steps in our lattice. Associate to each facet the binary string where 0 indicates a rightward step and 1 a vertical step. Then order the facets based on the lexicographic order on these strings, starting with $0\cdots01\cdots1$. For any facet of $F_i \times \Delta^k$, the vertices of the restriction face are all of the upper left corner vertices (the elements after an upward step and before a rightward step) along with the left-most element of each of the top i rows (some of these may also be upper-left corners). In the original BCS construction, each restriction face was only the upper left corner vertices. Hence for $F_0 \times \Delta^k$ we have the same restriction faces as in the BCS ordering, while for $F_i \times \Delta^k$ with $i > 1$ we are potentially including more vertices in each restriction face.

We show that these are in fact the restriction faces of a shelling order. First we show that each claimed restriction face is a new face. If we take a facet $G \in F_i \times \Delta^k$, its restriction face $r(G)$ cannot be in any previous facet in the shelling order of $F_i \times \Delta^k$ since by the BCS result we know that a (possibly proper) face of $r(G)$ is not in any previous facet. Furthur, $r(G)$ contains a vertex from each of the top i rows of F_i. By our definition of the F_i, no facet F_j with $j < i$ contains all of the vertices corresponding to these rows, so $r(G)$ cannot be contained in $F_j \times \Delta^k$ for $j < i$.

Second we show that for each vertex $v \in r(G)$ the face $G - \{v\}$ is in some previous facet in our shelling order. If v corresponds to an upper-left corner then this follows from the BCS result. Otherwise, v is the only element in one of the top i rows of the path corresponding to G, and hence $G - \{v\}$ is contained in a facet of some $F_j \times \Delta^k$ with $j < i$. Note that this last claim fails in the case $i = j + 1$, where it is possible for v to be neither an upper-left corner nor the only element in its row.

Now we count the size of the restriction face of each facet in $F_i \times \Delta^k$. When we consider the path corresponding to a facet of $F_i \times \Delta^k$, each time we reach a new row or column for the first time we must decide whether to go up or to the right (we include the first column but not the last column, and we exclude the bottom row). We must choose to go right a total of k times. Regardless of our choices, the top i rows each contribute one element to our restriction face. The other rows contribute an element exactly when we choose to go to the right upon reaching that row. Our choices for the columns do not influence the number of elements in the restriction face. We have $j + 1$ rows and $k + 1$ columns, hence the number of ways of getting a restriction face of size b is

$$\binom{j-i}{b-i}\binom{k+i}{k-(b-i)} = \binom{j-i}{b-i}\binom{k+i}{b}.$$

Since the inclusion of F_i only changes the h-vector of $\partial\Delta^{j+1}$ by adding one to h_i, we have the desired coefficients for the $h_m(A^j)$ in equation (4.4) for $m \leq j$.

For the case $m = j + 1$, we want to add in the final facet of $\partial\Delta^{j+1}$. The resulting product is not shellable, so we cannot use the above argument. However, in this step all we are adding to $\partial\Delta^{j+1}$ is a single j-face, so by Proposition 4.1 the change in h_b when we add in the product $F_{j+1} \times \Delta^k$ is

$$\binom{d+1-0}{d+1-b}\binom{0-1}{b-j-1} = \binom{k+(j+1)}{b}\binom{j-(j+1)}{b-(j+1)},$$

giving the desired coefficient of $h_{j+1}(A^j)$ in equation (4.4). $\square$

4.3 Changes in the g-vector Due to Surgery

In this section we use Theorem 4.2 to compute some of the interesting changes in the g-vector that occur when performing a surgery involving a product with a simplex. We consider two cases depending on whether we start with a simplex as our initial ball or the boundary of a simplex as our sphere.

Theorem 4.3. *Let B^j be a triangulated ball of dimension j and let $k > j$. Then*

$$g_{j+1}(B^j \times \partial\Delta^k) - g_{j+1}(\partial B^j \times \Delta^k) = \binom{j+k+1}{j+1}.$$

Proof. From Theorem 4.2 we have

$$g_{j+1}(B^j \times \partial\Delta^k) = \sum_{m=0}^{j+1}\left[\binom{j-m}{j+1-m}\binom{k+m}{j+1} - \binom{d+1-m}{d+j}\binom{m-1}{j-k}\right] h_m(B^j).$$

Since $k > j$ the binomial coefficient $\binom{m-1}{j-k}$ is always zero. For $m \leq j$ we know that $\binom{j-m}{j+1-m}$ is always zero. Hence the coefficients for $m \leq j$ in the above sum are all zero. Since B^j is a ball its reduced Euler characteristic is zero, so $h_{j+1}(B^j) = 0$. Therefore $g_{j+1}(B^j \times \partial\Delta^k) = 0$.

Again applying Theorem 4.2 (and noting that ∂B^j has dimension $j-1$) yields

$$\begin{aligned} g_{j+1}(\partial B^j \times \Delta^k) &= h_{j+1}(\partial B^j \times \Delta^k) - h_j(\partial B^j \times \Delta^k) \\ &= \sum_{m=0}^{j}\left[\binom{j-1-m}{j+1-m}\binom{k+m}{j+1} - \binom{j-1-m}{j-m}\binom{k+m}{j}\right] h_m(\partial B^j). \end{aligned}$$

For all $m < j$ the coefficient of $h_m(\partial B^j)$ is zero. Furthur, since ∂B^j is a sphere we know $h_j(\partial B^j) = 1$. Hence we have

$$\begin{aligned} g_{j+1}(\partial B^j \times \Delta^k) &= \left[\binom{-1}{1}\binom{k+j}{j+1} - \binom{-1}{0}\binom{k+j}{j}\right] h_j(\partial B^j) \\ &= \left[-\binom{k+j}{j+1} - \binom{k+j}{j}\right] \\ &= -\binom{k+j+1}{j+1}. \end{aligned}$$

Combining these two calculations gives the desired result. □

Theorem 4.4. *Let B^j be a triangulated ball of dimension j and let $j > k$. Then*

$$g_{k+1}(\partial B^j \times \Delta^k) - g_{k+1}(B^j \times \partial\Delta^k) = \binom{j+k+1}{k+1} + \sum_{i=1}^{k+1} \left(c_i g_i(B^j) + d_i g_i(\partial B^j)\right) \tag{4.5}$$

where

$$\begin{aligned} c_i &= \sum_{p=1}^{i-1} \left[\binom{j-p}{j-1-k}\binom{k+p}{k+1} - \binom{j+k-(p-1)}{j}\right] \quad \textit{and} \\ d_i &= \binom{j-i}{k+1-i}\binom{k+i}{k+1}. \end{aligned}$$

Note that by Lemma 4.9 the c_i are all non-positive.

Proof. Let $d = j + k$. Applying Theorem 4.2 and the fact that $h_{j+1}(B^j) = 0$ we have

$$\begin{aligned} g_{k+1}(B^j \times \partial\Delta^k) &= \sum_{m=0}^{j+1} \left[\binom{j-m}{k+1-m}\binom{k+m}{k+1} - \binom{d+1-m}{d-k}\binom{m-1}{0}\right] h_m(B^j) \\ &= \sum_{m=0}^{j} \left[\binom{j-m}{j-k-1}\binom{k+m}{k+1} - \binom{d+1-m}{j}\right] h_m(B^j). \end{aligned} \tag{4.6}$$

Again using Theorem 4.2,

$$\begin{aligned} g_{k+1}(\partial B^j \times \Delta^k) &= h_{k+1}(\partial B^j \times \Delta^k) - h_k(\partial B^j \times \Delta^k) \\ &= \sum_{m=0}^{j} \left[\binom{j-1-m}{k+1-m}\binom{k+m}{k+1} - \binom{j-1-m}{k-m}\binom{k+m}{k}\right] h_m(\partial B^j). \end{aligned} \tag{4.7}$$

Therefore, we combine equation (4.6) and the fact that $h_i(\Delta) = \sum_{j=0}^{i} g_j(\Delta)$ for any simplicial complex Δ to obtain

$$c_i = -\sum_{m=i}^{j} \left[\binom{j-m}{j-k-1}\binom{k+m}{k+1} - \binom{d+1-m}{j} \right].$$

The negative sign is because $B^j \times \partial\Delta^k$ is the original product that we are replacing. Comparing this equation with Lemma 4.9 we see that $c_1 = 0$.

For $i > 1$ the coefficient of $h_{i-1}(B^j)$ in equation (4.6) is equal to $c_i - c_{i-1}$. Because $c_1 = 0$, for $i > 1$ we have

$$c_i = \sum_{m=1}^{i-1} \left[\binom{j-m}{j-k-1}\binom{k+m}{k+1} - \binom{d+1-m}{j} \right],$$

as claimed in the theorem.

Applying the fact $h_i(\Delta) = \sum_{j=0}^{i} g_j(\Delta)$ to equation (4.7) we have

$$d_i = \sum_{m=i}^{j} \left[\binom{j-1-m}{k+1-m}\binom{k+m}{k+1} - \binom{j-1-m}{k-m}\binom{k+m}{k} \right].$$

By Lemma 4.10, these coefficients match the expression in the theorem. Also note that the c_i and d_i are all zero for $i > k+1$; hence it is valid for the upper limit of the sum in (4.5) to be $i = k+1$.

For $i = 0$ we get $d_i = 0$, so there is no contribution to the constant term in equation (4.5) from $g_{k+1}(\partial B^j \times \Delta^k)$. Since $c_1 = 0$, the contribution to the constant term from $g_{k+1}(B^j \times \partial\Delta^k)$ is the negative of the coefficient of $h_0(B^j)$ in equation (4.6). Hence our constant is

$$-\left[\binom{j}{j-k-1}\binom{k}{k+1} - \binom{d+1}{j} \right] = \binom{d+1}{j} = \binom{d+1}{k+1},$$

as desired. $\square$

4.4 A Restriction on the g-vectors of Manifolds

We begin by formally stating Kalai's conjectured lower bounds on the g-vector of a triangulated manifold. These lower bounds depend on the Betti numbers of the manifold. To express these bounds we use two generalizations of the h-vector known as the h'- and h''-vectors. For a homology $(d-1)$-manifold Δ we define

$$h'_i(\Delta) := h_i(\Delta) + \binom{d}{i} \sum_{j=1}^{i-1} (-1)^{i-j-1} \beta_{j-1}(\Delta).$$

The h'-vector is a combination of the h-vector and the Betti numbers of the manifold. In the case of a sphere it is the same as the standard h-vector. For more general manifolds, the h'-vector has the same algebraic interpretation as the h-vector of a sphere. Namely, Schenzel [31] showed that if Δ is a homology $(d-1)$-manifold and Θ is a linear system of parameters for $k[\Delta]$, then $\dim_k(k[\Delta]/\Theta)_i = h'_i(\Delta)$.

We define the h''-vector by $h''_d(\Delta) = h'_d(\Delta)$ and for $0 \leq i \leq d-1$

$$h''_i(\Delta) := h'_i(\Delta) - \binom{d}{i} \beta_{i-1}(\Delta) = h_i(\Delta) - \binom{d}{i} \sum_{j=1}^{i} (-1)^{i-j} \beta_{j-1}(\Delta).$$

For any homology $(d-1)$-manifold the entries of the h''-vector are non-negative and symmetric. Using the h'' vector, we state Kalai's conjecture [12, Conjecture 14.2, typo corrected].

Conjecture 4.5. *Let Δ be a triangulated $(d-1)$-manifold. Then for $0 \leq k \leq \lfloor d/2 \rfloor - 1$, $h''_{k+1}(\Delta) - h''_k(\Delta) \geq \binom{d}{k} \beta_k(\Delta)$.*

From the definition of the h''-vector we can rewrite these lower bounds as $g_{k+1} \geq \binom{d+1}{k+1}\beta_k + C$ where C depends on the Betti numbers β_i with $i < k$. The conjecture is known to hold when the links of the vertices of the manifold satisfy an algebraic version of the g-conjecture, for example when all of the vertex links are polytopal [28, Equation (9)].

From Theorem 4.4 we know that a surgery that replaces $B^j \times \partial\Delta^k$ with $\partial B^j \times \Delta^k$ results in a change in g_{k+1} by the (expected) constant $\binom{j+k+1}{k+1}$ plus $\sum_{i=1}^{k+1} (c_i g_i(B^j) + d_i g_i(\partial B^j))$ where the c_i are non-positive and the d_i are non-negative. For some balls B^j this sum has a strictly negative value (an example is discussed below). Consider the case where a surgery using such a ball increases the kth Betti number of our manifold by one. The new manifold that we construct by performing such a surgery must satisfy Kalai's conjectured (or proved, depending on the manifold) lower bound on g_{k+1}. This places a more restrictive condition on the g-vector of the original manifold than is obtained by applying the lower bound directly to the manifold.

For example, consider the case where our surgery is handle addition. In this situation Kalai's conjectured bound for $h_2'' - h_1''$ follows for all manifolds from an already-known portion of the g-conjecture. Let M be a triangulated $(d-1)$-manifold that contains two disjoint copies of a ball B^{d-1} such that the sum $\sum_{i=1}^{k+1} \left(c_i g_i(B^{d-1}) + d_i g_i(\partial B^{d-1})\right)$ is strictly negative. Then M admits a handle addition where $B^{d-1} \times \partial\Delta^1$ is replaced by $\partial B^{d-1} \times \Delta^1$. The existence of this handle addition places a more restrictive bound on $g_2(M)$.

In the case of handle addition ($k = 1$, $j = d-1$ in Theorem 4.4) our formula for c_i reduces to

$$c_i = \sum_{p=1}^{i-1} \left[\binom{d-1-p}{d-3}\binom{1+p}{2} - \binom{d+1-p}{d-1} \right].$$

We know $c_1 = 0$ and by direct calculation $c_2 = (d-2)\cdot 1 - (d) = -2$.

For the d_i, we again use Theorem 4.4 to obtain

$$d_i = \binom{d-1-i}{2-i}\binom{1+i}{2}.$$

Hence $d_1 = d-2$ and $d_2 = 3$. The total change in g_{k+1} after the handle addition is therefore

$$\binom{d+1}{2} - 2g_2(B^{d-1}) + (d-2)g_1(\partial B^{d-1}) + 3g_2(\partial B^{d-1}).$$

Combining this discusion with Kalai's bounds we have the following result.

Proposition 4.6. *Let $d \geq 4$ and let Δ be a connected homology $(d-1)$-manifold that contains two disjoint copies of a $(d-1)$-ball B. Then*

$$h_2''(\Delta) - h_1''(\Delta) \geq d \cdot \beta_1 + 2g_2(B) - (d-2)g_1(\partial B) - 3g_2(\partial B).$$

Using known constructions, there exist many balls B such that this lower bound is strictly greater than $d \cdot \beta_1$. For example, any ball with h-vector $(1, 2, 3, 3, \ldots, 3, 3, 2, 0)$ will have a boundary sphere with all h-vector entries equal to one.

4.5 Binomial Coefficient Identities

Lemma 4.7. *Let a, j and i be non-negative integers. Then*

$$\sum_{i=0}^{a} \binom{j+1-m}{i+1-m} \binom{a}{a-k, a-i, i+k-a} = \binom{a}{a-k} \binom{j+1-m+k}{a-m+1}. \tag{4.8}$$

Proof. Consider a collection of $j+1-m+k$ elements divided into two parts, one of size a (call this Y) and the other of size $j+1-m$ (call this Z). The right-hand side of (4.8) counts the number of ways of choosing a set P of $a-k$ elements of Y and a second set R of $a-m+1$ of the remaining elements (in Y or Z).

The left-hand side of (4.8) counts the same thing except partitioned by the index i, where $i+1$ is the number of items in the set R taken from Z. For each i, the multinomial coefficient counts the number of ways of diving Y into a set of size $a-i$ (to be included in R), a set of size $a-k$ (this is P), and a set of size $i+k-a$ (elements in neither P or R). Then from Z we choose the remaining $i+1-m$ elements for R. $\square$

Lemma 4.8. *Let a and c be non-negative integers and let $a+b-c \geq 0$. Then*

$$\sum_{i=c}^{a+b}(-1)^i \binom{a}{i-b}\binom{i}{c} = (-1)^{a+b}\binom{b}{c-a}.$$

Proof. Using standard binomial identities and re-indexing we have

$$\begin{aligned}
&= \sum_{i=c}^{a+b}(-1)^i \binom{a}{i-b}\binom{i}{i-c} \\
&= \sum_{i=c}^{a+b}(-1)^i \binom{a}{i-b}(-1)^{i-c}\binom{-i+(i-c)-1}{i-c} \\
&= (-1)^c \sum_{i=c}^{a+b} \binom{a}{a+b-i}\binom{-c-1}{i-c} \\
&= (-1)^c \sum_{i=0}^{a+b-c} \binom{a}{a+b-c-i}\binom{-c-1}{i}.
\end{aligned}$$

We now apply Vandermonde's convolution and simplify to obtain the desired result.

$$\begin{aligned}
\sum_{i=c}^{a+b}(-1)^i \binom{a}{i-b}\binom{i}{c} &= (-1)^c \binom{a-c-1}{a+b-c} \\
&= (-1)^c(-1)^{a+b-c}\binom{-(a-c-1)+(a+b-c)-1}{a+b-c} \\
&= (-1)^{a+b}\binom{b}{a+b-c} \\
&= (-1)^{a+b}\binom{b}{c-a}.
\end{aligned}$$

□

Lemma 4.9. *Let j and k be positive integers with $j > k$. For $1 \leq p \leq j$*

$$\sum_{m=1}^{p}\binom{j-m}{j-k-1}\binom{k+m}{k+1} \leq \sum_{m=1}^{p}\binom{j+k-(m-1)}{j},$$

with equality when $p = j$.

Proof. We prove the result by giving a combinatorial interpretation of both sums. We then define a bijection between the elements corresponding to the right-hand sum with $p = j$

and the elements corresponding to the left-hand sum with $p = j$. The index of each left-hand side element (its m value) will always be greater than or equal to the index of the corresponding right-hand side element, proving the desired inequalities.

We begin by describing our combinatorial objects. Each element of the combinatorial interpretation is a subset of j items chosen from $j + k$ items. On the left-hand side, this is done by choosing $j - (k+1)$ of the first $j - m$ items and then $k+1$ of the last $k+m$ items. For the right-hand side we first choose j items from an ordered set of $(j + k) - (m - 1)$ items. We make this into a choice of j items from $j + k$ items by adding $m - 1$ unselected items immediately before the $(j - k)$th selected item.

We now define our bijection. For an element on the left-hand side of index $m = n'$, define $n \in \mathbb{N}$ by setting $n + (j - n')$ to be the position of the $(j - k)$th item in the corresponding subset of $j + k$. In other words, n is the number of positions you must go beyond $j - n'$ to reach the $(j - k)$th item in the set. We map our element to the right-hand side element with index $m = n$ that has the same corresponding subset of $j + k$ items.

For an element on the right-hand side with index $m = n$, let q be the position of the $(j - k)$th item of the corresponding subset of $j+k$. Define n' such that $j-n' = (q-1)-(n-1)$, so $n' = j - q + n$. We map our element to the left-hand side element with index $m = n'$ that has the same corresponding subset of $j + k$ items.

It is straightforward to check that these two maps are well-defined and are inverses. Furthur, since we are selecting a subset of j items from $j + k$ items, the position q of the $(j - k)$th item is never greater than j. Hence we get $n' = j - q + n \geq n$, as desired. $\square$

Lemma 4.10. *Let j, k and p be positive integers with $j > k$ and $j \geq p$. Then*

$$\sum_{m=p}^{j} \left(\binom{j-1-m}{k+1-m}\binom{k+m}{k+1} - \binom{j-1-m}{k-m}\binom{k+m}{k} \right) = \binom{j-p}{k+1-p}\binom{k+p}{k+1}.$$

Proof. The proof is by induction on p with p decreasing. Both sides are zero when $p > k+1$ and the base case $p = k+1$ follows directly from the formula. Assuming the result for $m = p+1$ and using standard binomial identities we have

$$
\begin{aligned}
&\sum_{m=p}^{j}\left(\binom{j-1-m}{k+1-m}\binom{k+m}{k+1} - \binom{j-1-m}{k-m}\binom{k+m}{k}\right) \\
&= \left(\binom{j-1-p}{k+1-p}\binom{k+p}{k+1} - \binom{j-1-p}{k-p}\binom{k+p}{k}\right) + \binom{j-(p+1)}{k+1-(p+1)}\binom{k+(p+1)}{k+1} \\
&= \binom{j-1-p}{k+1-p}\binom{k+p}{k+1} + \binom{j-1-p}{k-p}\binom{k+p}{k+1} \\
&= \binom{j-p}{k+1-p}\binom{k+p}{k+1},
\end{aligned}
$$

completing the inductive step. $\square$

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